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TWINING CHARACTERS, ORBIT LIE ALGEBRAS, AND FIXED POINT RESOLUTION [T]

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Abstract.
We describe the resolution of field identification fixed points in coset conformal field theories in terms of representation spaces of the coset chiral algebra. A necessary ingredient from the representation theory of Kac–Moody algebras is the recently developed theory of twining characters and orbit Lie algebras, as applied to automorphisms representing identification currents.

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1 Fixed point resolution

While in the title of this paper we refer to some specific mathematical structures, namely certain (recently developed) aspects of the representation theory of Kac–Moody Lie algebras, the problem that we address is a rather general one, and in fact the basic ideas can be described without going into any technical details. This problem, to which we refer as *fixed point resolution*, arises whenever one has to mod out redundancies in the description of a system ('gauge symmetries') in a situation where the orbits of the redundancy transformations have unequal sizes, and hence potentially arises in various different areas of physics and mathematics.

The presence of a redundancy symmetry means that distinct combinations of the, a priori basic, data of a theory actually describe the same physical situation. Thus the physical state of the system in question is not described by an individual set of those data, but rather by an equivalence class of such sets that is given by an orbit of the redundancy symmetries. For instance, in Yang–Mills theory physical states do not correspond to individual configurations of the fundamental fields, but rather to gauge orbits thereof. In short, the prescription is

$$\boxed{\text{individual configurations } \Phi} \quad \longrightarrow \quad \boxed{\text{orbits } [\Phi]} \tag{1.1}$$

The prescription (1.1) must, however, possibly be modified as soon as orbits of different sizes are present. Such a situation occurs e.g. in Yang–Mills theory, where reducible connections lead to 'shorter' orbits, implying that the space of gauge orbits is not a smooth manifold, but a stratified variety which has singularities. While a priori it may well be consistent to work with the space of orbits even in this more complicated situation, one should not be surprised if doing so one encounters problems; e.g. in the Yang–Mills case one may think of ambiguities in the quantization procedure.

Here we consider a situation where the dynamics is under control so that the inconsistency of a naive implementation of the redundancies can be easily detected. Namely, we analyze coset theories, i.e. conformal field theories whose Virasoro algebra is obtained by subtracting from the Sugawara energy-momentum tensor of a WZW theory based on an affine Lie algebra g the Sugawara tensor of the WZW theory based on a subalgebra g' of g. The redundancy symmetry in this case includes, but is not exhausted by, the subalgebra g'. In the presence of orbits of different sizes, in these theories one would not obtain a modular invariant partition function when proceeding according to the prescription (1.1). One advantage of the type of theory we consider is that the presence of different orbit sizes can be attributed to the action of a discrete finite group.

The resolution of this problem is to supplement the recipe (1.1) by a 'resolution of the fixed points'. By this we simply mean that the states of the theory must be organized in a more complicated way than suggested by (1.1). This somewhat vague statement can be made explicit in the conformal field theory context, where it means that some of the naively obtained modules of the coset chiral algebra are *not* irreducible. Rather, the irreducible modules are submodules which are eigenspaces of the additional redundancy transformations. For the full theory obtained by combining its two chiral halves this implies that not all states one would naively expect are present, i.e. that on the space of states an additional projection takes place.

We defer a more precise description of these issues to section 7. Before that, we have to describe certain mathematical structures which are basic ingredients for the analysis of section 7. The key concepts are *twining characters* and *orbit Lie algebras* which have been introduced in [1]; these structures are also interesting in their own right.

2 Kac–Moody algebras and diagram automorphisms

Both twining characters and orbit Lie algebras arise in a much broader context than the one of affine Lie algebras that is relevant to the application in conformal field theory; they are both determined by two basic data:
(1) a symmetrizable Kac–Moody algebra g, and (2) a diagram automorphism ω of g.

A *symmetrizable Kac–Moody algebra* is a Lie algebra which possesses both a Cartan matrix and a Killing form. That is, there is a square matrix A, the Cartan matrix of g, which has diagonal entries $A^{i,i} = 2$ and non-positive integral off-diagonal entries such that $A^{i,j} = 0$ iff $A^{j,i} = 0$, as well as an invertible diagonal matrix D such that DA is symmetric. The algebra g associated to A is obtained as follows [2]. The Cartan subalgebra $\mathsf{g}_\circ$ is by definition an abelian Lie algebra of dimension $2n-r$, where n is the dimension and r the rank of A; there are (uniquely up to isomorphism) n linearly independent elements H^i of $\mathsf{g}_\circ$ and n linearly independent functionals $\alpha^{(i)} \in \mathsf{g}_\circ^\star$ (the simple roots) such that $\alpha^{(i)}(H^j) = A^{i,j}$ for $i,j = 1,2,\dots,n$; g is generated freely by $\mathsf{g}_\circ$ and by elements $E^i_\pm \equiv E^{\pm\alpha^{(i)}}$, with $i \in \{1,2,\dots,n\}$, modulo the relations

$$\begin{aligned} &[x,y] = 0 \quad \text{for all } x,y \in \mathsf{g}_\circ\,, && [x,E^i_\pm] = \pm\alpha^{(i)}(x)\,E^i_\pm \quad \text{for all } x \in \mathsf{g}_\circ\,, \\ &[E^i_+,E^j_-] = \delta_{ij}\,H^j\,, && (\mathrm{ad}_{E^i_\pm})^{1-A^{j,i}}\,E^j_\pm = 0 \quad \text{for } i \neq j\,. \end{aligned} \tag{2.1}$$

The Kac–Moody algebra g has a triangular decomposition $\mathsf{g} = \mathsf{g}_+ \oplus \mathsf{g}_\circ \oplus \mathsf{g}_-$, where $\mathsf{g}_\pm$ are subalgebras generated freely by the $E^i_\pm$ modulo the Serre relations, i.e. the last type of relations in (2.1). For any null eigenvector of the Cartan matrix, the corresponding linear combination of the H^i is a central element, and hence the central subalgebra g_K of g has dimension $n-r$. The derived algebra $\hat{\mathsf{g}} := [\mathsf{g},\mathsf{g}]$ of g contains g_K, and it has a triangular decomposition, namely $\hat{\mathsf{g}} = \mathsf{g}_+ \oplus \hat{\mathsf{g}}_\circ \oplus \mathsf{g}_-$, where $\hat{\mathsf{g}}_\circ$ is the span of the elements H^i, $i = 1,2,\dots,n$. The generators of a complement g_D of $\hat{\mathsf{g}}_\circ$ in $\mathsf{g}_\circ$ are called the derivations of g.

The Dynkin diagram of g is the graph with n vertices that has coincidence matrix $2\cdot\mathbb{1}-DA$. Without loss of generality, we will assume that the Dynkin diagram is connected, i.e. that A is indecomposable. A *diagram automorphism* ω of g is an automorphism that acts on the generators corresponding to simple roots like

$$\omega(E^i_\pm) := E^{\dot\omega i}_\pm \tag{2.2}$$

by a symmetry of the Dynkin diagram, i.e. by a permutation $\dot\omega$ satisfying $A^{\dot\omega i,\dot\omega j} = A^{i,j}$ for all i,j. The mapping (2.2) extends to automorphisms of g_+ and g_-, and by $\omega(H^i) := \omega([E^i_+,E^i_-]) = [E^{\dot\omega i}_+,E^{\dot\omega i}_-] = H^{\dot\omega i}$ it extends to a unique automorphism of $\hat{\mathsf{g}}$ of finite order.

The extension of ω to the rest of g is in general not unique. To see this, we choose a basis of $n-r$ eigenvectors K^a, $a = 1,2,\dots,n-r$, of the central subspace g_K such that

$$\omega(K^a) = \zeta^{n_a} K^a \qquad \text{with} \quad \zeta := \exp(2\pi\mathrm{i}/N) \tag{2.3}$$

(N is the order of $\dot\omega$); this is possible because ω maps g_K bijectively to itself. This extends to a basis of $\hat{\mathsf{g}}_\circ$ with further generators J^p, $p = 1,2,\dots,r$, satisfying $\omega(J^p) = \zeta^{m_p} J^p$. The restriction of the non-degenerate invariant bilinear form $(\cdot\mid\cdot)$ on $\mathsf{g}_\circ$ to the span g_J of the J^p is non-degenerate, and $(K^a \mid x) = 0$ for all $x \in \hat{\mathsf{g}}_\circ$. It follows that there are $n-r$ unique elements D^a of $\mathsf{g}_\circ$ such that $(D^a \mid K^a) = 1$ and $(D^a \mid \cdot\,) = 0$ for all other basis elements of $\mathsf{g}_\circ$. The D^a

form a basis for a complement g_D of $\hat{\mathsf{g}}_\circ$ in $\mathsf{g}_\circ$. Using the automorphism property of ω, we find

$$\omega(D^a) = \zeta^{-n_a} D^a + \sum_{b=1}^{n-r} M^a_b \zeta^{n_b} K^b \,, \tag{2.4}$$

with M an antisymmetric $(n-r)\times(n-r)$ matrix.

Thus the freedom in defining ω consists in adding central terms to $\omega(D^a)$ and is parametrized by the antisymmetric matrix M. In particular, if g is simple, affine, or hyperbolic, ω is in fact uniquely determined by $\dot\omega$. Also, when acting on $\hat{\mathsf{g}}$, ω has the same order N as $\dot\omega$, and hence in the general case we can (and will) restrict the freedom in $\omega(D)$ further by requiring that ω has order N on all of g; then M^a_b must vanish whenever $n_a = -n_b \bmod N$. It is quite important that the only freedom in the action of ω consists in adding central terms. In the highest weight representations of our interest, the central elements act as multiples of the identity; in particular, the character-like quantities we will consider are affected by the freedom in choosing ω only via multiplication with an overall factor.

3 Twining characters

The ordinary character χ_V of a g-module V on which the action of the Cartan subalgebra can be diagonalized is a (formal) function on the Cartan subalgebra $\mathsf{g}_\circ$, defined by

$$\chi_V : \quad \mathsf{g}_\circ \to \mathbb{C} \,, \quad \chi_V(h) := \mathrm{tr}_V \mathrm{e}^{2\pi\mathrm{i} R_\Lambda(h)} \,. \tag{3.1}$$

(R is the representation in which g acts on V; from now on we suppress the symbol R whenever no confusion can arise.) The character can be expanded in terms of the weights λ of V as $\chi_V = \sum_\lambda m_\lambda \, \mathrm{e}^{2\pi\mathrm{i}\lambda}$, where m_λ is the multiplicity of λ, i.e. the dimension of the subspace $W_{(\lambda)}$ of V that has weight λ. Hence χ_V is the generating functional for the dimensions of the weight spaces. For highest weight modules V_Λ with highest weight Λ, $m_\lambda \neq 0$ implies that $\lambda \le \Lambda$, i.e. that $\Lambda-\lambda$ is a linear combination of simple roots with non-negative integral coefficients.

Now the idea of the twining character is to count not just multiplicities of states, but in addition to keep track of the action of ω, analogous as for a character-valued index. To put this idea to work, we must specify what 'the action of ω' on V means. This map, to be denoted by τ_ω, is the natural induced action on the g-module (V,R), which means that τ_ω does not change V as an abstract vector space while it does change the representation $R\colon\ \mathsf{g} \to \mathit{End}(V)$ of g by endomorphisms $R(x)$ of V; namely, the representation of g on V changes according to the map

$$R(x) \;\mapsto\; R^\omega(x) := R(\omega(x)) \tag{3.2}$$

for all $x \in \mathsf{g}$. Thus in general the map does change the (isomorphism class of the) module. [1] Further, the action of ω preserves the triangular decomposition of g so that (V, R^ω_Λ) is again a Verma module if (V,R_Λ) is. Moreover, the sets of primitive singular vectors of (V,R_Λ) and (V,R^ω_Λ) coincide, and hence acting on the irreducible quotient of a Verma module yields again an irreducible module. To save space, for the time being we will denote both the Verma module with highest weight Λ and its irreducible quotient by the same symbol V_Λ.

[1] In other words, $R(\mathsf{g})$ and $R^\omega(\mathsf{g})$ describe two, generically inequivalent, embeddings of g into the algebra $\mathit{End}(V)$. It is only after applying to (V,R) and (V,R^ω) the forgetful functor from the category of g-modules to the category of vector spaces that (V,R) and (V,R^ω) become identical objects.

While the highest weight vector in the modules (V,R) and (V,R^ω) is one and the same element $v_{\rm h.w.}$ of the underlying vector space V, its associated weight gets transformed by the dual action $\omega^\star$ of ω on the weight space ${\sf g}_\circ^\star$ of ${\sf g}$ (this mapping $\beta\mapsto\omega^\star\beta$ reads $(\omega^\star\beta)(x)=\beta(\omega^{-1}x)$ for all $\beta\in{\sf g}_\circ^\star$ and all $x\in{\sf g}_\circ$). Thus, while $v_{\rm h.w.}$ has weight Λ in (V,R_Λ), it has weight $\omega^\star\Lambda$ in (V,R_Λ^ω). In other words, (V,R_Λ) is isomorphic to the module V_Λ, while (V,R_Λ^ω) is isomorphic to $V_{\omega^\star\Lambda}$. Via these isomorphisms one and the same vector $v\in V$ is identified with an element v' of V_Λ and another element v'' of $V_{\omega^\star\Lambda}$. We therefore define τ_ω as the map $\tau_\omega\colon\ V_\Lambda\to V_\Lambda^\omega$ that acts as $v'\mapsto v''$; then $\tau_\omega(R_\Lambda(x)\cdot v)=R_{\omega^\star\Lambda}(\omega(x))\cdot\tau_\omega(v)$, i.e. the diagram

$$\begin{array}{ccc} V_\Lambda & \xrightarrow{R_\Lambda(x)} & V_\Lambda \\ {\scriptstyle\tau_\omega}\downarrow & & \downarrow{\scriptstyle\tau_\omega} \\ V_{\omega^\star\Lambda} & \xrightarrow{R_{\omega^\star\Lambda}(\omega(x))} & V_{\omega^\star\Lambda} \end{array} \tag{3.3}$$

commutes for all $x\in{\sf g}$ and all $v\in V_\Lambda$. Thus τ_ω behaves as a generalization of an intertwiner map; accordingly we refer to (3.3) as the *ω-twining property* of τ_ω.

The proper implementation of the idea to keep track of the action of ω in the character is thus expressed by the following definition. The *twining character* (or automorphism-twined character) $\mathcal{X}_\Lambda^{[\omega]}$ of a (Verma or irreducible) module V_Λ is the (formal) function

$$\mathcal{X}_\Lambda^{[\omega]}:\quad {\sf g}_\circ\to\mathbb{C}\,,\qquad \mathcal{X}_\Lambda^{[\omega]}(h):={\rm tr}_{V_\Lambda}\tau_\omega\,{\rm e}^{2\pi{\rm i}R_\Lambda(h)} \tag{3.4}$$

on ${\sf g}_\circ$. Analogously as for the ordinary character $\mathcal{X}$, we can write the twining character as

$$\mathcal{X}_\Lambda^{[\omega]}=\sum_{\lambda\le\Lambda}m_\lambda^{[\omega]}\,{\rm e}^{2\pi{\rm i}\lambda}\,. \tag{3.5}$$

Here $m_\lambda^{[\omega]}$ is non-zero only if $\omega^\star(\lambda)=\lambda$, in which case it is the trace of the restriction of τ_ω to the weight space $W_{(\lambda)}$ of weight λ; thus the twining character is the generating functional for these traces. Strictly speaking, the notation 'tr' makes sense only if $\omega^\star\Lambda=\Lambda$, since only in this case τ_ω is an endomorphism of a single highest weight module rather than a map between two different ones. We will refer to such ${\sf g}$-weights Λ as *symmetric* weights, and simply define $\mathcal{X}_\Lambda^{[\omega]}$ to be zero for non-symmetric weights; below we will be mainly interested in symmetric weights. Generically some contributions to $\mathcal{X}_\Lambda^{[\omega]}$ have non-zero phase, so that is not at all obvious that the term character is an appropriate name for the object defined by (3.4). However, as we will see below, under suitable conditions the expansion coefficients $m_\lambda^{[\omega]}$ are non-negative integers, so that the term character is indeed justified.

The following properties of twining characters can be derived in a straightforward manner.

- The twining character is majorized by the ordinary character of the same module, and hence in particular its domain of convergence contains the one of the ordinary character.
- Since $m_\lambda^{[\omega]}\ne0$ implies $\omega^\star\lambda=\lambda$, we can restrict the sum in (3.5) to symmetric weights.
- Together with the cyclic invariance of the trace, the ω-twining property (3.3) implies that

$$\mathcal{X}_\Lambda^{[\omega]}(h)=\mathcal{X}_\Lambda^{[\omega]}(\omega(h)) \tag{3.6}$$

for all $h\in{\sf g}_\circ$.

- Using the eigenspace decompositions ${\sf g}_\circ=\bigoplus{\sf g}_\circ^{(j)}$ and ${\sf g}_\circ^\star=\bigoplus{\sf g}_\circ^{\star(j)}$, with $\omega x=\zeta^jx\ \forall\ x\in$

$\mathsf{g}_\circ^{(j)}$, $\omega^\star\beta = \zeta^j\beta \;\forall\; \beta\in\mathsf{g}_\circ^{\star(j)}$, it follows that the symmetric weights, i.e. those in $\mathsf{g}_\circ^{\star(0)}$, are non-zero only on the symmetric part $\mathsf{g}_\circ^{(0)}$ of the Cartan subalgebra. This implies that $\mathcal{X}_\Lambda^{[\omega]}(h) = \sum_\lambda m_\lambda^{[\omega]}\, \mathrm{e}^{2\pi\mathrm{i}\lambda}(\sum_j h^{(j)}) = \sum_\lambda m_\lambda^{[\omega]}\, \exp[2\pi\mathrm{i}\,\lambda(h^{(0)})]$, so that (3.6) can be strengthened to

$$\mathcal{X}_\Lambda^{[\omega]}(h) = \mathcal{X}_\Lambda^{[\omega]}(h^{(0)})\,. \tag{3.7}$$

Thus the twining character depends on $h\in\mathsf{g}_\circ$ non-trivially only through its component $h^{(0)}$ in $\mathsf{g}_\circ^{(0)}$. Correspondingly, from now on we will view $\mathcal{X}_\Lambda^{[\omega]}$ no longer as a function on $\mathsf{g}_\circ$, but on $\mathsf{g}_\circ^{(0)}$.

4 Orbit Lie algebras

To analyze the twining characters in more detail, we need another concept, namely the notion of an *orbit Lie algebra*. This concept is based on the simple observation that any symmetry $\dot\omega$ of a Dynkin diagram divides the set of nodes of the diagram into invariant subsets of size N_i, the orbits $[i]$ of $\dot\omega$. Now these orbits can be viewed as the nodes of another Dynkin diagram, obtained by an appropriate 'folding'. The correct prescription includes suitable weight factors; in terms of the associated Cartan matrix A, it reads as follows:

- sum up the N_i rows of A belonging to each orbit $[i]$;
- multiply the resulting row with the integer $s_i := 1-\sum_{l=1}^{N_i-1} A^{\dot\omega^l i,i} \in \mathbb{Z}_{>0}$;
- eliminate redundant columns.

Thus we obtain a matrix $\breve A$ whose rows and columns are labelled by the $\dot\omega$-orbits, with entries

$$\breve A^{[i],[j]} := s_i \frac{N_i}{N} \sum_{l=0}^{N-1} A^{\dot\omega^l i,j}\,. \tag{4.1}$$

In general, this is no longer a symmetrizable Cartan matrix. However, we can prove that $\breve A$ is again a symmetrizable Cartan matrix whenever

$$s_i \in \{1,2\} \quad \text{for all } i\,; \tag{4.2}$$

this requirement will be called the *linking condition*. In terms of the Dynkin diagram, (4.2) means that each of the nodes on an orbit is either connected by a single link to precisely one node on the same orbit or not linked at all to other nodes on the same orbit. In the sequel we will often restrict our attention to the class of those Dynkin diagram symmetries $\dot\omega$ which do satisfy the linking condition.

The proof that condition (4.2) implies that $\breve A$ is a symmetrizable Cartan matrix is not difficult. For instance, one has immediately $\breve A^{[i],[i]} = s_i\,(3-s_i) = 2$, and the integrality of the entries of $\breve A$ is made manifest by rewriting (4.1) as $\breve A^{[i],[j]} = s_i \sum_{l=0}^{N_i-1} A^{\dot\omega^l i,j}$. Also, if $D = \mathrm{diag}(d_i)$ is a non-singular diagonal matrix such that DA is symmetric, then $\breve D = \mathrm{diag}(\breve d_{[i]})$ with $\breve d_{[i]} := N d_i/s_i N_i$ is a non-singular diagonal matrix such that $\breve D\breve A$ is symmetric.

Symmetrizable Cartan matrices are either of *finite* (or *simple*) type (then the matrix DA is positive definite), of affine type (then DA is positive semidefinite and has exactly one eigenvector with eigenvalue zero), or else of *indefinite* type. One can see that the matrix $\breve A$ obtained from A by the prescription (4.1) is always of the same type as A. Also, by inspection one finds that all symmetries of simple and affine Dynkin diagrams satisfy the linking condition, except for the

order N automorphisms of the affine Lie algebras $A^{(1)}_{N-1}$ which rotate the Dynkin diagram (but the latter will also be dealt with separately below). Further, for the indefinite Cartan matrices which are *hyperbolic* (i.e., characterized by the property that any connected subdiagram of the Dynkin diagram that one obtains by deleting any node from the Dynkin diagram of g is finite or affine), again also $\breve{A}$ is hyperbolic.

We denote the Kac–Moody algebra corresponding to the Cartan matrix $\breve{A}$ by $\breve{\mathsf{g}}$ and call it the *orbit Lie algebra* associated to g and ω. A crucial observation is that the Cartan subalgebra $\breve{\mathsf{g}}_\circ$ of $\breve{\mathsf{g}}$ can be related to the subalgebra $\mathsf{g}_\circ^{(0)}$ of the Cartan subalgebra $\mathsf{g}_\circ$ of g that is invariant under ω by defining, for $h=\sum_i v_i H^i \in \hat{\mathsf{g}}_\circ^{(0)} \equiv \mathsf{g}_\circ^{(0)} \cap \hat{\mathsf{g}}_\circ$,

$$\mathrm{P}_\omega(h) := \sum_{[i]} N_i v_i\, \breve{H}^{[i]} \tag{4.3}$$

(this does not depend on the choice of representatives of the orbits $[i]$, because $v_i = v_{\dot\omega^l i}$ for $h \in \hat{\mathsf{g}}_\circ^{(0)}$). The map P_ω has the following properties:

- The mapping $\mathrm{P}_\omega\colon\ \hat{\mathsf{g}}_\circ^{(0)} \to \hat{\breve{\mathsf{g}}}_\circ$ is one-to-one.
- The invariant bilinear form on $\hat{\breve{\mathsf{g}}}_\circ$ is proportional to the restriction to $\hat{\mathsf{g}}_\circ^{(0)}$ of the invariant bilinear form on $\hat{\mathsf{g}}_\circ$: for all $h_1, h_2 \in \hat{\mathsf{g}}_\circ^{(0)}$ we have

$$(\mathrm{P}_\omega(h_1) \mid \mathrm{P}_\omega(h_2)) = N\,(h_1 \mid h_2)\,. \tag{4.4}$$

- If $K \in \hat{\mathsf{g}}_\circ^{(0)}$ is central, then so is $\mathrm{P}_\omega(K) \in \hat{\breve{\mathsf{g}}}_\circ$, and vice versa. Thus in particular the dimension of the space $\mathsf{g}_K^{(0)}$ of invariant central elements is the number of $\dot\omega$-orbits minus the rank of $\breve{A}$.
- As a consequence, one can employ the non-degeneracy of the invariant bilinear form on $\mathsf{g}_\circ^{(0)}$ to extend the map P_ω to all of $\mathsf{g}_\circ$ in such a way that (4.4) is still valid.
- By duality, the correspondence between $\mathsf{g}_\circ^{(0)}$ and $\breve{\mathsf{g}}_\circ$ implies analogous relations for the weight spaces. Thus there is a bijective map $\mathrm{P}_\omega^\star\colon\ \breve{\mathsf{g}}_\circ^\star \to \mathsf{g}_\circ^{\star(0)}$ between the weights of $\breve{\mathsf{g}}$ and the symmetric weights $\lambda \in \mathsf{g}_\circ^{\star(0)}$, such that, analogously to (4.4), $(\lambda \mid \mu) = N\cdot(\mathrm{P}_\omega^{\star-1}(\lambda) \mid \mathrm{P}_\omega^{\star-1}(\mu))$. For brevity, we will often denote the pre-image $\mathrm{P}_\omega^{\star-1}(\lambda) \in \breve{\mathsf{g}}_\circ^\star$ of a symmetric weight λ simply by $\breve{\lambda}$.

5 Twining characters as characters of orbit Lie algebras

The two structures introduced above – twining characters and orbit Lie algebras – are both determined through a symmetrizable Kac–Moody algebra g and a diagram automorphism ω of g, so that one should not be surprised that they are closely related. Nevertheless we think that the precise form of the relationship is most remarkable. Namely, provided that ω satisfies the linking condition, we find, both for Verma modules and for their irreducible quotient modules:

The twining characters of the highest weight modules over g coincide with the ordinary characters of the highest weight modules over the orbit Lie algebra $\breve{\mathsf{g}}$.

To specify what 'coincide' means, the relevant 'projections' needed for this statement to make sense must be performed, i.e. more explicitly the assertion of this theorem is that

$$\boxed{\mathcal{X}_\Lambda^{[\omega]}(h) = \breve{\mathcal{X}}_{\breve\Lambda}(\breve h) \qquad \text{with} \qquad \breve\Lambda = \mathrm{P}_\omega^{\star-1}(\Lambda) \quad \text{and} \quad \breve h = \mathrm{P}_\omega(h)\,.} \tag{5.1}$$

An immediate consequence of this identification is that the term 'character' for $\mathcal{X}_\Lambda^{[\omega]}$ is indeed justified; also, under suitable circumstances $\mathcal{X}_\Lambda^{[\omega]}$ has nice modular transformation properties.

As already mentioned, the linking condition is not satisfied for the order N diagram automorphisms of $A^{(1)}_{N-1}$ which do appear in the application to conformal field theory. But we can treat these cases as well, and the result is again very simple: the only non-zero term in the expansion (3.5) for both the irreducible and Verma modules is the one for the highest weight, i.e. we have

$$m^{[\omega]}_\lambda = 0 \quad \text{for } \lambda \neq \Lambda \,. \tag{5.2}$$

The proof of the statements (5.1) and (5.2) is a mixture of a 'conceptual proof' and of 'verification' by explicit calculations. We will concentrate on the conceptual parts; the computational details can be found in [1]. A crucial ingredient is the identification of a natural action of $\breve W$, the Weyl group of the orbit Lie algebra $\breve{\mathsf g}$, on the twining characters.

■ We claim that the Weyl group of $\breve{\mathsf g}$ is isomorphic to a subgroup of the Weyl group of $\mathsf g$, and that the action of this subgroup on $\mathsf g^\star_\circ$ commutes with the action of $\omega^\star$.

We denote by w_i and $\breve w_{[i]}$ the fundamental reflections which generate the Weyl groups W of $\mathsf g$ and $\breve W$ of $\breve{\mathsf g}$, respectively, and employ the map $\mathrm P^\star_\omega$ to push the action of $\breve W$ on the weight space $\breve{\mathsf g}^\star_\circ$ of the orbit Lie algebra $\breve{\mathsf g}$ to an action on $\mathsf g^{\star(0)}_\circ$. First we show that for any $\breve w_{[i]} \in \breve W$ there exists an associated element $\hat w_{[i]}$ of $\hat W$ which acts on $\mathsf g^{\star(0)}_\circ$ precisely like $\breve w_{[i]}$ acts on $\breve{\mathsf g}^\star_\circ$, i.e. $\mathrm P^{\star-1}_\omega(\hat w_{[i]}(\lambda)) = \breve w_{[i]}(\mathrm P^{\star-1}_\omega(\lambda))$ for all $\lambda \in \mathsf g^{\star(0)}_\circ$. Now by direct calculation, we obtain $\mathrm P^\star_\omega(\breve\alpha^{(i)\vee}) = \frac{N}{N_i}\sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)\vee}$ and $\mathrm P^\star_\omega(\breve\alpha^{(i)}) = s_i\sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)}$, which implies that the latter relation is equivalent to requiring that the action of $\hat w_{[i]}$ reads

$$\hat w_{[i]}(\lambda) = \lambda - s_i \cdot \sum_{l=0}^{N_i-1} (\lambda \mid \alpha^{(\dot\omega^l i)\vee})\, \alpha^{(\dot\omega^l i)} \,. \tag{5.3}$$

We denote the mapping $\breve w_{[i]} \mapsto \hat w_{[i]}$ by $\mathrm P_W$. As can be checked by employing the relations in the Weyl group W, the ansatz

$$\hat w_{[i]} = \left\{ \begin{array}{ll} \prod_{l=0}^{N_i-1} w_{\dot\omega^l i} & \text{for } s_i = 1\,, \\ w_i\, w_{\dot\omega i}\, w_i = w_{\dot\omega i}\, w_i\, w_{\dot\omega i} & \text{for } N_i = s_i = 2 \end{array} \right. \tag{5.4}$$

does fulfill (5.3). [2] (The relations of W also imply that for $s_i = 1$ the multiple product in (5.4) does not depend on the order of the factors.)

We can also check that the $\hat w_{[i]}$ are reflections, i.e. square to the identity. Moreover, $\hat w_{[i]}$ commutes with $\omega^\star$, $[\hat w_{[i]}, \omega^\star] = 0$ for all i; this implies in particular that the action of $\hat w_{[i]}$ respects the orbits of $\omega^\star$. Next we define $\hat W$ as the subgroup of W that is generated by the elements $\hat w_{[i]}$. To show that the map $\breve w_{[i]} \mapsto \hat w_{[i]}$ extends to a group isomorphism $\mathrm P_W\colon\ \breve W \to \hat W$, we view the Weyl group $\breve W$ as the Coxeter group that is freely generated by the generators $\breve w_{[i]}$ modulo the relations $(\breve w_{[i]})^2 = id$ and $(\breve w_{[i]}\breve w_{[j]})^{\breve m_{[i],[j]}} = id$ for all i, j with $i \neq j$, where $\breve m_{[i],[j]} = 2, 3, 4, 6$ for $\breve A^{[i],[j]}\breve A^{[j],[i]} = 0, 1, 2, 3$ and (formally) $\breve m_{[i],[j]} = \infty$ for $\breve A^{[i],[j]}\breve A^{[j],[i]} \geq 4$. We can prove that $\hat W$ obeys identical relations, i.e. is a Coxeter group with $\hat m_{[i],[j]} = \breve m_{[i],[j]}$, and hence $\hat W \cong \breve W$.

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[2] For $s_i = 2$, there is precisely one $m \in \{1, \ldots, N_i - 1\}$ with $A^{\dot\omega^m i, i} = -1$. It follows that $\dot\omega^m i = \dot\omega^{-m} i$, which in turn implies that the orbit length N_i is even and that the restriction of the Dynkin diagram of $\mathsf g$ to this orbit is the Dynkin diagram of $N_i/2$ copies of A_2. As a consequence, without loss of generality we can restrict ourselves to the case $N_i = 2$. Otherwise we first treat the automorphism $\dot\omega^{N_i/2}$, which has order two and possesses $N_i/2$ orbits each of which corresponds to the Dynkin diagram of A_2. On the set of orbits of $\dot\omega^{N_i/2}$, the automorphism $\dot\omega$ induces an automorphism $\ddot\omega$ of order $N_i/2$; all orbits with respect to $\ddot\omega$ have $s_j = 1$.

What is in fact easy to see is that $\hat{m}_{[i],[j]} \geq \breve{m}_{[i],[j]}$. Namely, if $(\hat{w}_{[i]}\hat{w}_{[j]})^{\hat{m}_{[i],[j]}} = id \in W$, then in particular $(\hat{w}_{[i]}\hat{w}_{[j]})^{\hat{m}_{[i],[j]}}$ acts as the identity on $\mathsf{g}_\circ^{\star(0)}$; then also $(\breve{w}_{[i]}\breve{w}_{[j]})^{\hat{m}_{[i],[j]}} \in \breve{W}$ acts as the identity on the weight space of $\breve{\mathsf{g}}$, hence is the identity element of $\breve{W}$; this implies that $\breve{m}_{[i],[j]}$ must be a divisor of $\hat{m}_{[i],[j]}$. The inequality $\hat{m}_{[i],[j]} \geq \breve{m}_{[i],[j]}$ already proves the assertion for $\breve{A}^{[i],[j]}\breve{A}^{[j],[i]} \geq 4$. In the remaining cases one can show, by exploiting the explicit form of the Coxeter relations for W, that also $\hat{m}_{[i],[j]} \leq \breve{m}_{[i],[j]}$, so that equality follows again. However, this necessitates a lengthy case by case study depending on the value of $A^{i,j}A^{j,i}$ (and also of s_i, s_j, N_i, N_j) [1]; the details can be found in [1].

■ The next step consists in considering the action of the group $\hat{W}$ on the twining characters that is induced via the action (5.3) on g-weights.

From this point on the proof proceeds in a way rather similar to Kac' proof of the Weyl–Kac character formula, see e.g. [2, pp. 152, 172], though the technical details (which we omit) are somewhat more complicated, which is related to the necessary distinction between the cases $s_i = 1$ and $s_i = N_i = 2$ [1]. In particular we must now carefully distinguish between Verma and irreducible modules. Thus in the sequel we will use the previous notations V_Λ and $\mathcal{X}_\Lambda^{[\omega]}$ only for Verma modules and their twining characters, while we write L_Λ for the irreducible quotient of V_Λ and $\chi_\Lambda^{[\omega]}$ for the twining character of L_Λ.

■ For Verma modules, we find that the combination

$$\mathcal{X}^{[\omega]} := \mathrm{e}^{-\rho-\Lambda}\mathcal{X}_\Lambda^{[\omega]} \tag{5.5}$$

(which is independent of Λ and corresponds to interpreting the Verma module as the universal enveloping algebra of g_-) is odd under the action (5.3) of $\hat{W}$,

$$\hat{w}(\mathcal{X}^{[\omega]}) = \hat{\epsilon}(\hat{w})\, \mathcal{X}^{[\omega]}\,. \tag{5.6}$$

Here the sign function

$$\hat{\epsilon}(\hat{w}) := \breve{\epsilon}(\mathrm{P}_W^{-1}(\hat{w})) \tag{5.7}$$

is the homomorphism $\hat{\epsilon}$ from $\hat{W}$ to $\mathbb{Z}_2$ that is the pull-back of the sign function $\breve{\epsilon}$ on $\breve{W}$, rather than the sign function that $\hat{W}$ inherits as a subgroup from the sign function ϵ of W.

■ For the action of $\hat{W}$ on the twining characters of irreducible modules we obtain

$$\hat{w}(\chi_\Lambda^{[\omega]}) = \chi_\Lambda^{[\omega]} \tag{5.8}$$

for all $\hat{w} \in \hat{W}$, provided that the highest weight Λ is dominant integral; i.e., the twining character of an irreducible highest weight module with dominant integral highest weight is $\hat{W}$-even.

Proof:

■ To enter the proof for Verma modules, we choose a specific basis $\mathcal{B}_-$ of g_-, including a suitable enumeration of the elements of $\mathcal{B}_-$. For $s_i = 1$, we choose as the first N_i elements of $\mathcal{B}_-$ the step operators $E_-^{\dot{\omega}^l i}$ for $l = 0, 1, ..., N_i - 1$ and then the step operators associated to all other negative roots in an arbitrary ordering. Then according to the Poincaré–Birkhoff–Witt theorem the set of all products $\mathcal{E}^{(\vec{n},\vec{m})} = \mathcal{E}_1^{(\vec{n})} \cdot \mathcal{E}_2^{(\vec{m})}$ with $\mathcal{E}_1^{(\vec{n})} := (E^{-\alpha^{(i)}})^{n_0}\,(E^{-\alpha^{(\dot{\omega}i)}})^{n_1} \ldots (E^{-\alpha^{(\dot{\omega}^{N_i-1}i)}})^{n_{N_i-1}}$ and $\mathcal{E}_2^{(\vec{m})} := (E^{-\beta_1})^{m_1}(E^{-\beta_2})^{m_2}\ldots$, with n_i and m_i non-negative integers only finitely many of which are different from zero, forms a basis of $\mathsf{U}(\mathsf{g}_-)$. Commutator terms that arise when reshuffling the products of generators of g_- do not contribute to $\mathcal{X}^{[\omega]}$, since $\mathcal{X}^{[\omega]}$ is a trace, and an element $v^{(\vec{n},\vec{m})} = \mathcal{E}^{(\vec{n},\vec{m})} \cdot v_\Lambda$ of the Verma module can contribute to $\mathcal{X}^{[\omega]}$ only if $n_0 = n_1 = \ldots = n_{N_i-1} =: n$. The PBW theorem also implies that the contributions to $\mathcal{X}^{[\omega]}$ stemming from the products $\mathcal{E}_1^{(\vec{n})}$ and $\mathcal{E}_2^{(\vec{m})}$ factorize, so that we can investigate their transformation properties under $\hat{w}_{[i]}$ separately. First, $\hat{w}_{[i]}$ commutes with

$\omega^\star$ and maps any $\omega^\star$-orbit of negative roots to some other orbit of negative roots, i.e. only permutes the orbits that contribute to the second factor, and hence this factor is invariant under $\hat{w}_{[i]}$. On the other hand, the contribution $(\mathcal{X}^{[\omega]})_1$ of operators of the type $\mathcal{E}_1^{(\vec{n})}$ to $\mathcal{X}^{[\omega]}$ can be computed explicitly as $(\mathcal{X}^{[\omega]})_1 = \mathrm{e}^{-\rho}/(1-\exp[-\sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)}])$. Also, evaluating (5.3) we find

$$\hat{w}_{[i]}(\alpha^{(\dot\omega^l i)}) = w_{\dot\omega^l i}(\alpha^{(\dot\omega^l i)}) = -\alpha^{(\dot\omega^l i)}\,, \tag{5.9}$$

as well as

$$\hat{w}_{[i]}(\rho) = \rho - s_i {\textstyle\sum_{l=0}^{N_i-1}}\alpha^{(\dot\omega^l i)} \tag{5.10}$$

for any Weyl vector $\rho\in\mathsf{g}^\star$, i.e. any weight with $\rho(H^i)=1$ for all i. We then obtain $\hat{w}_{[i]}((\mathcal{X}^{[\omega]})_1) = \exp[-\rho+\sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)}]/(1-\exp[\sum_{l=0}^{N_i-1}\alpha^{(\dot\omega^l i)}]) = -(\mathcal{X}^{[\omega]})_1$, and hence, combining the two factors, $\hat{w}_{[i]}(\mathcal{X}^{[\omega]}) = -\mathcal{X}^{[\omega]}$. This holds for all generators $\hat{w}_{[i]}$, and hence we arrive at (5.6).

In the case $s_i = N_i = 2$ we choose a different basis $\mathcal{B}_-$ of g_-. As the first three elements of $\mathcal{B}_-$ we take the step operators $E^{-\alpha^{(i)}}$, $E^{-\alpha^{(\dot\omega i)}}$ and $E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}}$, and then again the step operators corresponding to all other negative roots in an arbitrary ordering. A basis of $\mathsf{U}(\mathsf{g}_-)$ is then given similar as above, but with the first factor $\mathcal{E}_1^{(\vec{n})}$ replaced by $\tilde{\mathcal{E}}_1^{(n_0,n_1,n')} := (E^i_-)^{n_0}(E^{\dot\omega i}_-)^{n_1}(E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}})^{n'}$. The contribution to $\mathcal{X}^{[\omega]}$ from operators of the type $\mathcal{E}_2^{(\vec{m})}$ again transforms trivially under $\hat{w}_{[i]}$. Further, in order to have a contribution $(\mathcal{X}^{[\omega]})_1$ from operators of the type $\tilde{\mathcal{E}}_1^{(n_0,n_1,n')}$, we need again $n_0 = n_1 =: n$. The transformation properties $\omega(E^i_-) = E^{\dot\omega i}_-$ and $\omega(E^{\dot\omega i}_-) = E^i_-$ imply that $\omega(E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}}) = -E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}}$, which allows us to compute the contribution $(\mathcal{X}^{[\omega]})_1$ to the twining character as $(\mathcal{X}^{[\omega]})_1 = \mathrm{e}^{-\rho}(1-\mathrm{e}^{-2\alpha^{(i)}-2\alpha^{(\dot\omega i)}})^{-1}$. Also, from (5.3) we have

$$\hat{w}_{[i]}(\alpha^{(i)}) = -\alpha^{(\dot\omega i)}\,,\quad \hat{w}_{[i]}(\alpha^{(\dot\omega i)}) = -\alpha^{(i)}\,,\quad \hat{w}_{[i]}(\alpha^{(i)}+\alpha^{(\dot\omega i)}) = -(\alpha^{(i)}+\alpha^{(\dot\omega i)})\,, \tag{5.11}$$

while any other positive root is again mapped on a positive root different from $\alpha^{(i)}$, $\alpha^{(\dot\omega i)}$ and $\alpha^{(i)}+\alpha^{(\dot\omega i)}$. With (5.10) it then follows again that this contribution to $\mathcal{X}^{[\omega]}$ changes sign under the action of $\hat{w}_{[i]}$, and hence again we obtain (5.6).

■ For irreducible modules the basic idea is to consider the decomposition $L_\Lambda = \bigoplus L_{(L_k)}$ of the irreducible module L_Λ into irreducible modules of g_i, where

$$\mathsf{g}_i := \left\{ \begin{array}{ll} \langle E_\pm^{\dot\omega^l i}, H^{\dot\omega^l i} \mid l=0,1,\ldots,N_i-1\rangle & \text{for } s_i=1\,, \\ \langle E^{\pm\alpha^{(i)}\pm\alpha^{(\dot\omega i)}},\, H^i+H^{\dot\omega i}\rangle & \text{for } s_i=N_i=2\,. \end{array}\right. \tag{5.12}$$

Among the modules $L_{(L_k)}$, only those contribute to $\chi_\Lambda^{[\omega]}$ which are mapped to themselves by τ_ω. For $s_i=1$, where g_i is isomorphic to a direct sum of N_i copies of A_1 algebras, only those basis vectors in these modules contribute to $\chi_\Lambda^{[\omega]}$ which have the same weight with respect to all the A_1 ideals, $\ell_1=\ell_2=\ldots=\ell_{N_i}$. These weights are all flipped in sign by $\omega^\star$, and hence τ_ω maps a basis vector $v=v_\ell$ to a vector v' proportional to $v_{-\ell} = (E^i_-E^{\dot\omega i}_-\ldots E^{\dot\omega^{N_i-1}i}_-)^l\,v$; by the ω-twining property (3.3) it then follows that the eigenvalue equation $\tau_\omega(v)=\zeta^k v$ implies $\tau_\omega(v')=\zeta^k(v')$, i.e. v and v' contribute the same phase to $\chi_\Lambda^{[\omega]}$. As this is true for all states, it follows that $\hat{w}_{[i]}(\chi_\Lambda^{[\omega]}) = \chi_\Lambda^{[\omega]}$, and as this is true for all values of i, we have (5.8).

In the case $s_i=N_i=2$, where g_i is isomorphic to A_1, there is a slight complication, because the automorphism ω acts on g_i as $\omega(E^{\pm\alpha^{(i)}\pm\alpha^{(\dot\omega i)}}) = -E^{\pm\alpha^{(i)}\pm\alpha^{(\dot\omega i)}}$, $\omega(H^i+H^{\dot\omega i}) = H^i+H^{\dot\omega i}$. However, we now have $v' \propto (E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}})^{2l}v$, where the factor of two arises because the H^i- and $H^{\dot\omega i}$-eigenvalues are added up. Thus only even powers of the step operator $E^{-\alpha^{(i)}-\alpha^{(\dot\omega i)}}$ occur, so that the additional minus signs cancel and the reasoning above applies again.

■ Next we derive a linear relation between irreducible and Verma twining characters.

If $\mu \le \lambda$, i.e. if $\lambda-\mu = \sum_i n_i \alpha^{(i)}$ with $n_i \in \mathbb{Z}_{\ge 0}$, then we call $\mathrm{dp}_\lambda(\mu) := \sum_i n_i$ the depth of μ with respect to λ. By induction on the depth, we can show that

$$\mathcal{X}_\lambda^{[\omega]} = \textstyle\sum_{\mu\le\lambda} \tilde c_{\lambda\mu}\, \chi_\mu^{[\omega]} \,, \tag{5.13}$$

with $\mathcal{X}_\lambda^{[\omega]}$ the twining character of the Verma module with arbitrary symmetric highest weight λ and $\chi_\mu^{[\omega]}$ the twining character of the irreducible module with highest weight μ; the numbers $\tilde c_{\lambda\mu}$ are contained in the cyclotomic field extension $\mathbb{Q}(\zeta)$ of the rationals and obey $\tilde c_{\lambda\lambda}=1$.

In order to contribute to the sum (5.13), the weights μ must obey further requirements in addition to $\mu\le\lambda$. In particular μ must be symmetric, and hence we can restrict the summation to symmetric weights μ for which $\breve\mu \le \breve\lambda$. Further, the generalized second order Casimir operator of g has the constant value $C_2(\lambda) = (\lambda+2\rho\,|\,\lambda)$ on V_λ; then by (4.4) also $|\breve\mu+\breve\rho|$ has a fixed value, namely $|\breve\lambda+\breve\rho|$, for all weights μ which contribute to (5.13). (Also note that for any Weyl vector ρ of g the $\breve{\mathsf{g}}$-weight $\breve\rho \equiv \mathrm{P}_\omega^{\star-1}(\rho)$ is a Weyl vector of $\breve{\mathsf{g}}$.) In short, in the decomposition (5.13) we can restrict μ to the subset

$$\hat B(\lambda) := \{\, \mu = \mathrm{P}^\star_\omega(\breve\mu) \mid \breve\mu\le\breve\lambda,\ |\breve\mu+\breve\rho| = |\breve\lambda+\breve\rho| \,\} \,. \tag{5.14}$$

Furthermore, for any dominant integral highest weight Λ we can label the elements of $\hat B(\Lambda)$ as λ_i with $i\in\mathbb{N}$, in such a way that $\breve\lambda_j \le \breve\lambda_i$ implies $i\le j$. Applying the analogue of (5.13) to all elements of $\hat B(\Lambda)$, it then follows that for all λ_i we have $\mathcal{X}_{\lambda_i}^{[\omega]} = \sum_{\lambda_j\in\hat B(\Lambda)} \tilde c_{ij}\, \chi_{\lambda_j}^{[\omega]}$ with coefficients $\tilde c_{ij} \in \mathbb{Q}(\zeta)$ which satisfy $\tilde c_{ii}=1$. Further, $\tilde c_{ij}$ can be non-zero only if $\breve\lambda_j \le \breve\lambda_i$, so that the matrix $\tilde c = (\tilde c_{ij})$ is upper triangular and can be inverted. Its inverse $c=(c_{ij})$ is upper triangular as well and obeys $c_{ii}=1$, and hence the twining character $\chi_\Lambda^{[\omega]}$ of the irreducible module with highest weight Λ can be written as an (infinite) linear combination $\chi_\Lambda^{[\omega]} = \sum_{\lambda\in\hat B(\Lambda)} c_\lambda\, \mathcal{X}_\lambda^{[\omega]}$, or in terms of the universal Verma twining character $\mathcal{X}^{[\omega]}$,

$$\chi_\Lambda^{[\omega]} = \mathcal{X}^{[\omega]} \cdot \sum_{\lambda\in\hat B(\Lambda)} c_\lambda\, \mathrm{e}^{\lambda+\rho} \,. \tag{5.15}$$

■ Finally we compare (5.15) with the behavior of the twining characters under $\hat W$.
Since $\mathcal{X}^{[\omega]}$ is odd under the action of $\hat W$ while the left hand side of (5.15) is $\hat W$-even, the sum on the right hand side must be $\hat W$-odd, so that $c_\lambda = \hat\epsilon(\hat w) c_\mu$ whenever $\hat w(\lambda+\rho) = \mu+\rho$ for some element $\hat w\in\hat W$. Thus for all $\hat w\in\hat W$ we have $c_\lambda = \hat\epsilon(\hat w)\, c_{\hat w(\lambda+\rho)-\rho}$. Moreover, with any weight λ the weight system of L_Λ already contains the full $\hat W$-orbit of λ, and hence we need to know c_λ only for a single element of each $\hat W$-orbit, or, equivalently, for a single element of each $\breve W$-orbit of weights of the $\breve{\mathsf{g}}$-module $L_{\breve\Lambda}$, say the unique weight in the fundamental Weyl chamber of $\breve{\mathsf{g}}$. But the only such weight $\breve\lambda$ which obeys both $\breve\lambda \le \breve\Lambda = \mathrm{P}_\omega^{\star-1}(\Lambda)$ and $|\breve\lambda+\breve\rho| = |\breve\Lambda+\breve\rho|$ is the highest weight $\breve\Lambda$ itself; thus only a single Weyl orbit contributes,

$$\chi_\Lambda^{[\omega]} = \mathcal{X}^{[\omega]} \cdot \sum_{\hat w\in\hat W} \hat\epsilon(\hat w)\, \mathrm{e}^{\hat w(\Lambda+\rho)} \,. \tag{5.16}$$

Evaluating this identity for the trivial one-dimensional irreducible module with highest weight $\Lambda=0$, for which $\chi_0^{[\omega]}=1$, we have $\mathcal{X}^{[\omega]} \equiv (\sum_{\hat w\in\hat W} \hat\epsilon(\hat w)\, \mathrm{e}^{\hat w(\rho)})^{-1}$, so that

$$\chi_\Lambda^{[\omega]} = \sum_{\hat w\in\hat W} \hat\epsilon(\hat w)\, \mathrm{e}^{\hat w(\Lambda+\rho)} \Big/ \sum_{\hat w\in\hat W} \hat\epsilon(\hat w)\, \mathrm{e}^{\hat w(\rho)} \,. \tag{5.17}$$

When applied to $h \in \mathsf{g}_\circ^{(0)}$, this can be rewritten as

$$\chi_\Lambda^{[\omega]}(h) = \frac{\sum_{\breve w\in\breve W} \breve\epsilon(\breve w)\, \mathrm{e}^{(\breve w(\breve\Lambda+\breve\rho))(\mathrm{P}_\omega h)}}{\sum_{\breve w\in\breve W} \breve\epsilon(\breve w)\, \mathrm{e}^{(\breve w(\breve\rho))(\mathrm{P}_\omega h)}} \,, \tag{5.18}$$

and hence

$$\chi_\Lambda^{[\omega]}(h) = \chi_{\breve\Lambda}(\mathrm{P}_\omega h) \tag{5.19}$$

by the usual Weyl–Kac character formula for integrable highest weight modules of $\breve{\mathsf{g}}$.

This completes the proof of our assertion (5.1) for the irreducible twining characters. Analogously, the statement for Verma twining characters follows by comparing the result for $\mathcal{X}^{[\omega]}$ with the formula for the Verma module characters of $\breve{\mathsf{g}}$ (since $\mathcal{X}^{[\omega]}$ is independent of Λ, this latter result holds for arbitrary highest weights, not just for dominant integral ones).

6 Affine algebras

For the application to conformal field theory, we are interested in the special case where g is an untwisted affine Lie algebra and where ω is a diagram automorphism that corresponds to a simple current of the associated WZW theory. [3] In this case we can make a number of additional observations.

■ In the realization of untwisted affine Lie algebras as centrally extended loop algebras with a derivation, a basis of g is given by H^i_m and $E^{\bar\alpha}_m$ together with the canonical central element K and a derivation, where $m\in\mathbb{Z}$, i takes values in the index set $\bar I$ that corresponds to the horizontal subalgebra $\bar{\mathsf{g}}$ of g, and $\bar\alpha$ is a root of $\bar{\mathsf{g}}$. The full index set is commonly written as $I := \bar I \cup \{0\} = \{0,1,2,...\,,r\}$; [4] in this notation, a simple current automorphism must obey $\dot\omega 0 \neq 0$.

The diagram automorphism ω acts as $\omega(K)=K$ and

$$\omega(H^i_n) = H^{\dot\omega i}_n \,, \qquad \omega(E^{\bar\alpha}_n) = \eta_{\bar\alpha}\, E^{\bar\omega^\star\bar\alpha}_{n+\ell_{\bar\alpha}} \,. \tag{6.1}$$

Here the bar refers to the horizontal subalgebra $\bar{\mathsf{g}}$, the prefactors $\eta_{\bar\alpha}$ are signs which are $+1$ for the simple roots and for all other roots are fixed by the automorphism property of ω , $\ell_{\bar\alpha} := (\bar\alpha, \bar\Lambda_{(\dot\omega^{-1}0)})$ with $\bar\Lambda_{(i)}$, $i=1,2,...\,,r$, the horizontal fundamental weights, and $\bar\omega^\star$ is an affine map on the weight space of $\bar{\mathsf{g}}$ acting as

$$\bar\omega^\star\bar\lambda = k^\vee_\lambda \bar\Lambda_{(\dot\omega 0)} + \sum_{\substack{j=1\\ j\neq\dot\omega 0}}^{r} \bar\lambda^{\dot\omega^{-1}j}\bar\Lambda_{(j)} - (\sum_{j=1}^{r} a^\vee_j \bar\lambda^j)\, \bar\Lambda_{(\dot\omega 0)} \,. \tag{6.2}$$

On the simple g-roots $\alpha^{(i)}$ and the fundamental g-weights $\Lambda_{(i)}$ $\omega^\star$ acts as $\omega^\star(\alpha^{(i)}) = \alpha^{(\dot\omega i)}$, $\omega^\star(\Lambda_{(i)}) = \Lambda_{(\dot\omega i)}$.

■ As the generator for the (one-dimensional) space g_D of derivations one commonly chooses the element L_0 defined by $[L_0, E^i_\pm] = \mp\delta_{i,0} E^0_\pm$ and $(L_0 \,|\, L_0) = 0$. The automorphism property

[3] These automorphisms can be characterized as the elements of the unique maximal abelian normal subgroup $\mathcal{Z}(\mathsf{g})$ of the group $\Gamma(\mathsf{g})$ of diagram automorphisms. This abelian subgroup is isomorphic to the center of the universal covering Lie group that has the horizontal subalgebra $\bar{\mathsf{g}} \subset \mathsf{g}$ as its Lie algebra.

[4] However, by construction the index set $\breve I$ for $\breve{\mathsf{g}}$ is then generically *not* the subset $\{0,1,...\,,\mathrm{rank}\,\bar{\breve{\mathsf{g}}}\}$ of $\{0,1,...\,,\mathrm{rank}\,\bar{\mathsf{g}}\}$.

of ω and the ω-invariance of the invariant bilinear form determine $\omega(L_0)$ uniquely, and there is in fact a unique extension of ω to the semi-direct sum of g and the Virasoro algebra, given by

$$\omega(L_m) = L_m - (\bar\Lambda_{(\dot\omega 0)}, H_m) + \tfrac{1}{2}\,(\bar\Lambda_{(\dot\omega 0)}, \bar\Lambda_{(\dot\omega 0)})\,\delta_{m,0}\,K \tag{6.3}$$

(and $\omega(C) = C$). Using (6.3), one can show that for any vector of a g-module whose weight is symmetric, the action of $\omega(L_0)$ coincides with that of L_0; this result is also a consequence of the Sugawara formula for the Virasoro generators.
The derivation D defined in the general case, which obeys $\omega(D) = D$, is related to L_0 by $D = -L_0 + N^{-1}\sum_{l=1}^{N-1}(\bar\Lambda_{(\dot\omega^l 0)}, H) - \frac{1}{2}\,N^{-2}\Gamma_{00}\,K$ with $\Gamma_{00} := \sum_{l,l'=1}^{N-1}(\bar\Lambda_{(\dot\omega^l 0)}, \bar\Lambda_{(\dot\omega^{l'} 0)})$.

The relation between the derivations $\breve L_0$ (defined analogously as L_0) and $\breve D = \mathrm{P}_\omega(D)$ of $\breve{\mathsf{g}}$ is $\breve D = -N\breve L_0$, which implies $\mathrm{P}_\omega(L_0) = N\breve L_0 + \mathrm{P}_\omega(\sum_{l=1}^{N-1}\bar\Lambda_{(\dot\omega^l 0)}, H)/N - \Gamma_{00}\breve K/2N^2$.

■ A sufficient (and, of course, necessary) condition for $\lambda \in \mathsf{g}_\circ^\star$ to be a symmetric weight is $\lambda^i = \lambda^{\dot\omega^l i}$ for all $i = 0, 1, \ldots, r$ and all l, i.e. there is no extra condition on the δ-component of λ.

■ Employing the explicit form of the action of $\omega^\star$ on the simple roots $\alpha^{(i)}$ and fundamental weights $\Lambda_{(i)}$ of g and the action of $\mathrm{P}_\omega^\star$ on the simple roots $\breve\alpha^{(i)}$ and fundamental weights $\breve\Lambda_{(i)}$ of $\breve{\mathsf{g}}$, one obtains a series of nice identities which can be used to simplify formulæ. For instance, for the metric of the horizontal part of $\breve{\mathsf{g}}^\star$ we can derive the relation

$$\bar{\breve G}_{ij} = (\bar{\breve\Lambda}_{(i)}, \bar{\breve\Lambda}_{(j)}) = \frac{1}{N}\,(\mathrm{P}_\omega^\star\bar{\breve\Lambda}_{(i)}, \mathrm{P}_\omega^\star\bar{\breve\Lambda}_{(j)}) = \frac{N_iN_j}{N^3}\,\Big[\sum_{l,l'=0}^{N-1}\bar G_{\dot\omega^l i,\dot\omega^{l'} j} - a_i^\vee a_j^\vee\Gamma_{00}\Big]\,. \tag{6.4}$$

Thus we have $(\bar\lambda, \bar\mu) = N\,(\bar{\breve\lambda}, \bar{\breve\mu}) + \Gamma_{00}\,\breve k_\lambda^\vee \breve k_\mu^\vee$ for $\breve{\mathsf{g}}$-weights $\breve\lambda$ and $\breve\mu$ at levels $\breve k_\lambda^\vee$ and $\breve k_\mu^\vee$, respectively.
■ In particular, the quadratic Casimir eigenvalue of a symmetric highest g-weight Λ at level $k^\vee$ can be written as $\bar C_2(\Lambda) \equiv (\bar\Lambda, \bar\Lambda + 2\bar\rho) = N\,(\bar{\breve\Lambda}, \bar{\breve\Lambda} + 2\bar{\breve\rho}) + \Gamma_{00}\,\breve k^\vee(\breve k^\vee + 2\breve g^\vee)$. Further, the dual Coxeter numbers $\breve g^\vee \equiv \sum_{i\in\breve I}\breve a_i^\vee$ of $\breve{\mathsf{g}}$ and $g^\vee \equiv \sum_{i=0}^{r} a_i^\vee$ of g are related by $N\,\breve g^\vee = g^\vee$, and for any symmetric g-weight λ of level $k_\lambda^\vee$, the level of $\breve\lambda = \mathrm{P}_\omega^{\star-1}(\lambda)$ is $\breve k_\lambda^\vee = k_\lambda^\vee/N$. This implies a simple relation between the conformal weights $\Delta_\Lambda \equiv \bar C_2(\Lambda)/((\bar\theta, \bar\theta)(k^\vee + g^\vee))$ of primary fields of the g and $\breve{\mathsf{g}}$ WZW theories (at levels $k^\vee$ and $\breve k^\vee$, respectively), namely

$$\Delta_\Lambda = \breve\Delta_{\breve\Lambda} + \tfrac{1}{2N^2}\,\Gamma_{00}k^\vee\,(1 + \tfrac{g^\vee}{k^\vee + g^\vee}) = \breve\Delta_{\breve\Lambda} + \tfrac{1}{24}\,\big[\tfrac{k^\vee}{g^\vee}\,(D - \breve D) + c - \breve c\big] \tag{6.5}$$

with D and $\breve D$ the dimension of $\bar{\mathsf{g}}$ and $\bar{\breve{\mathsf{g}}}$, respectively.
■ The order N automorphisms of the affine Lie algebras $A_{N-1}^{(1)}$ are generated by the permutation $\dot\omega$ that 'rotates the Dynkin diagram by one unit'. Thus all N nodes of the Dynkin diagram lie on a single $\dot\omega$-orbit, so that the level of any symmetric integrable highest weight is a multiple of N, and only a single such weight occurs at that level. The prescription (4.1) then formally yields the 1×1 'Cartan matrix' $A = 0$.
The ω-invariant subspace $\mathsf{g}_\circ^{(0)}$ of $\mathsf{g}_\circ$ is two-dimensional; it is spanned by the central element K and the derivation D. As K is central, we can therefore write

$$\mathcal{X}_\Lambda^{[\omega]}(t,\tau) \equiv \mathcal{X}_\Lambda^{[\omega]}(tK + \tau L_0) = \mathrm{e}^{2\pi\mathrm{i}tk_\Lambda}\cdot \mathrm{tr}_{V_\Lambda}\tau_\omega \mathrm{e}^{2\pi\mathrm{i}\tau L_0}\,. \tag{6.6}$$

Applying the Poincaré–Birkhoff–Witt theorem with a suitable basis for g_-, we can obtain a simple product formula for $\mathcal{X}_\Lambda^{[\omega]}$. By suitably rearranging factors (within finite products) in

this formula, we can then show that $\overline{\mathcal{X}}^{[\omega]}(q) \equiv \mathcal{X}_\Lambda^{[\omega]}(t=0, q=\mathrm{e}^{2\pi\mathrm{i}\tau})$ satisfies the functional equation $\overline{\mathcal{X}}^{[\omega]}(q) = \overline{\mathcal{X}}^{[\omega]}(q^N)$, which implies that $\overline{\mathcal{X}}^{[\omega]}(q)$ is constant. Evaluating the function at $q=0$, we find that $\overline{\mathcal{X}}^{[\omega]}(q) \equiv 1$, which means that only the highest weight contributes to the twining character of the Verma module. As the highest weight vector is never a null vector, this statement also applies to the irreducible module. Hence

$$\chi_\Lambda^{[\omega]}(t,\tau) = \mathcal{X}_\Lambda^{[\omega]}(t,\tau) = \mathrm{e}^{2\pi\mathrm{i}tk_\Lambda}\mathrm{e}^{2\pi\mathrm{i}\tau\Delta_\Lambda}\,, \tag{6.7}$$

with Δ_Λ the eigenvalue of L_0 on the highest weight vector. This proves our assertion (5.2).

■ The characters of $\breve{\mathsf{g}}$, and hence also the twining characters, have nice modular properties. At any fixed non-negative integral value of the level, the set of irreducible highest weight modules with dominant integral highest weights carries a unitary representation of SL(2,$\mathbb{Z}$). This group acts naturally on the modified characters $\tilde\chi_\Lambda \equiv \mathrm{e}^{-s_\Lambda\delta}\chi_\Lambda$, where s_Λ is the number

$$s_\Lambda := \frac{1}{(\bar\theta,\bar\theta)}\left(\frac{(\bar\Lambda+\bar\rho,\bar\Lambda+\bar\rho)}{k^\vee+g^\vee} - \frac{(\bar\rho,\bar\rho)}{g^\vee}\right) = \Delta_\Lambda - \frac{c}{24} \tag{6.8}$$

(with $c = k^\vee D/(k^\vee+g^\vee)$ the Virasoro central charge), called the modular anomaly.

In order to inherit the nice modular transformation properties from the modified characters of $\breve{\mathsf{g}}$, the *modified twining characters* must be defined as

$$\tilde\chi_\Lambda^{[\omega]} := \mathrm{e}^{-\hat s_\Lambda\hat\delta}\chi_\Lambda^{[\omega]} \tag{6.9}$$

with $\hat\delta = \mathrm{P}_\omega^\star(\breve\delta)$ and $\hat s_\Lambda := \breve s_{\mathrm{P}_\omega^{\star-1}\Lambda} \equiv \breve s_{\breve\Lambda} = \bar C_2(\breve\Lambda)/((\breve{\bar\theta},\breve{\bar\theta})(\breve k^\vee+\breve g^\vee)) - \breve c/24$. That is, the modification relevant to the twining character of g is *not* the one of the ordinary character of g, i.e. $\exp(-s_\Lambda\delta)$, but rather the pull-back of the modification of the $\breve{\mathsf{g}}$-character. Thus

$$\tilde\chi_\Lambda^{[\omega]}(h) = \exp\left[\breve s_{\mathrm{P}_\omega^{\star-1}\Lambda}(\mathrm{P}_\omega^\star\,\breve\delta)(h)\right]\breve\chi_{\breve\Lambda}(\mathrm{P}_\omega h) = \exp\left[\breve s_{\mathrm{P}_\omega^{\star-1}\Lambda}\,\breve\delta(\mathrm{P}_\omega h)\right]\breve\chi_{\breve\Lambda}(\mathrm{P}_\omega h) = \tilde{\breve\chi}_{\breve\Lambda}(\mathrm{P}_\omega h)\,. \tag{6.10}$$

However, we can show that in fact $\mathrm{P}_\omega^\star(\breve\delta) = \delta$, and hence because of (6.5) the required modification differs from the modification of the ordinary characters just by an overall constant which only depends on the level of Λ:

$$s_\Lambda = \Delta_\Lambda - \frac{c}{24} = \breve\Delta_{\breve\Lambda} - \frac{\breve c}{24} + \frac{1}{24}\frac{k^\vee}{g^\vee}(D-\breve D) = \hat s_\Lambda + \frac{1}{24}\frac{k^\vee}{g^\vee}(D-\breve D)\,. \tag{6.11}$$

■ For the simple current automorphisms of order two of $\mathsf{g} = C_{2n}^{(1)}$ or $B_{n+1}^{(1)}$, the orbit Lie algebra is the twisted affine Lie algebra $\tilde B_n^{(2)}$. [5] Our results fit with previous observations at odd level $k^\vee = 2\ell+1$ of g [4] which suggest a relation of $\breve{\mathsf{g}}$ to the $C_n^{(1)}$ WZW theory at level ℓ. Indeed, we can show that the modular S- and T-matrices of $\breve{\mathsf{g}} = \tilde B_n^{(2)}$ at level $\breve k^\vee = k^\vee$ coincide (up to sign factors which are related to certain flips arising in the application to fixed point resolution) with the S- and T-matrices of $C_n^{(1)}$ at level $\ell = (k^\vee-1)/2$. Also, for even levels our results prove a conjecture [5] for the S-matrix which was based on a level-rank duality of $N=2$ superconformal coset models.

[5] We use the notation of [3]; in [2], these algebras are denoted by $A_{2n}^{(2)}$.

7 Coset theories

We can now finally apply what we learned above to the situation outlined in the beginning, i.e. to coset conformal field theories.

The basic idea of the coset construction ([6], compare also [7]) is that, given a reductive Lie algebra $\bar{\mathsf{g}}$ and a reductive subalgebra $\bar{\mathsf{g}}' \hookrightarrow \bar{\mathsf{g}}$, one considers the corresponding embedding $\mathsf{g}' \hookrightarrow \mathsf{g}$ of the affinizations g (at some level $k^\vee \in \mathbb{Z}_{\geq 0}$) and g' and analyzes the difference

$$\dot{L}_m := L_m - L'_m \tag{7.1}$$

of the generators of the two Virasoro algebras which are associated to the affine Kac–Moody algebras g and g' via the Sugawara formula. The $\dot{L}_m$ generate again a Virasoro algebra, with central charge $\dot{c} = c - c'$, the *coset Virasoro algebra*. The basic questions are then the following.

- Does this prescription define a consistent conformal field theory (which then is referred to as the *coset theory* and briefly denoted by '$\bar{\mathsf{g}}/\bar{\mathsf{g}}'$')?
- If so, is that conformal field theory unique?
- What is the (maximally extended) chiral symmetry algebra $\dot{\mathcal{W}}$ of the coset theory?
- What is the spectrum of the theory, i.e. which irreducible highest weight modules (of $\dot{\mathcal{W}}$, or at least of the coset Virasoro algebra) appear?

While the coset construction has been proposed already a long time ago, to a large extent these crucial questions are still open.

Here we will provide an answer to the last question. (This will also have some impact on all the other questions, but it should be stressed that in particular the problem of describing the chiral algebra $\dot{\mathcal{W}}$ is not yet fully solved.) The main concepts needed to arrive at this answer are branching functions [2], field identification [8,9,10], simple currents [11,12,13,4], and fixed point resolution [4, 14, 15, 5, 16], and it is via the latter concept that twining characters and orbit Lie algebras come into the game [16].

In more mathematical terms, the question about the spectrum of the coset theory is the following. Given the coset Virasoro algebra associated to a specific pair $\bar{\mathsf{g}}$ and $\bar{\mathsf{g}}'$, what are the representation spaces on which it acts? By construction, the generators (7.1) are defined on the chiral state space of the WZW theory based on g, i.e. on the direct sum $\mathcal{H}_\mathsf{g} = \bigoplus_\Lambda L_\Lambda$ of all inequivalent irreducible highest weight modules at level $k^\vee$. However, as it turns out, this is *not* the state space of the coset theory. A first crucial observation is that g' commutes with all the generators $\dot{L}_n$. This implies that if we would retain the full state space $\mathcal{H}_\mathsf{g}$, then the coset theory would have spin zero fields other than the identity field, namely all the currents $J^{a'}(z)$ of the subalgebra g'. To avoid this disaster, we must require that these fields act trivially on the state space of the coset theory. This requirement is implemented by a *gauge principle*: any two states in $\mathcal{H}_\mathsf{g}$ which differ only by the action of g' are considered as different descriptions of the same physical situation, i.e. as representing one and the same vector in the state space of the coset theory.

We conclude that consistency requirements force us to consider g'-*orbits* of vectors in $\mathcal{H}_\mathsf{g}$ rather than individual vectors of $\mathcal{H}_\mathsf{g}$. Concretely, this is implemented by decomposing the g-modules L_Λ into g'-modules $L_{\Lambda'}$ as [6] $L_\Lambda = \bigoplus_{\Lambda'} L_{\Lambda,\Lambda'} \otimes L_{\Lambda'}$, and tentatively considering $L_{\Lambda,\Lambda'}$ as

[6] We denote quantities referring to the subalgebra g' by the same symbol as the corresponding quantities for g, but with a prime added. However, in order to simplify notation, we suppress 'double priming', e.g. do not attach a prime to an object that already has a primed subscript or superscript.

irreducible modules of the coset theory. In terms of characters, this corresponds to regarding the *branching functions* $b_{\Lambda,\Lambda'}$ which appear in the decomposition

$$\chi^{\mathsf{g}}_\Lambda(\tau) = \sum_{\Lambda'} b_{\Lambda,\Lambda'}(\tau)\, \chi^{\mathsf{g}'}_{\Lambda'}(\tau) \tag{7.2}$$

as the characters of the coset theory. However, when doing so severe problems still remain; they can be attributed to the fact that in general the redundancy symmetry of the theory is larger than the obvious gauge symmetry g'. First, typically there are selection rules, i.e. the branching functions for certain pairs $(\Lambda;\Lambda')$ vanish. This observation is in itself not too disturbing, as one just has to make sure to find all selection rules. (But still, while empirically for most coset theories all selection rules come from conjugacy class selection rules for the embedding $\bar{\mathsf{g}}' \hookrightarrow \bar{\mathsf{g}}$, so far a prescription to enumerate all selection rules for arbitrary coset theories is not known.) However, along with each selection rule there also comes a redundancy; namely, non-vanishing branching functions for distinct pairs $(\Lambda;\Lambda')$ turn out to be identical, in particular the putative vacuum module seems to occur several times. By the same argument which forced us to divide out the action of the subalgebra g', we see that this degeneracy cannot be interpreted as the multiple appearance of a corresponding primary field in the spectrum (in particular, it would then not be possible to obtain the required modular transformation properties). Rather, the correct interpretation is that a primary field of the coset theory is not associated to an individual pair $(\Lambda;\Lambda')$, but rather to an appropriate equivalence class $[\Lambda;\Lambda']$ of such pairs. This prescription has been termed *field identification*.

A very convenient description of both conjugacy class selection rules and field identification is provided by the concept of *identification currents* [14,4]. Namely, first, there is a subgroup $G_{\rm id}$ of the group of integer spin simple currents of the tensor product theory $\mathsf{g}\oplus(\mathsf{g}')^*$, the so-called identification group, such that the conjugacy class selection rules are given by the condition that the monodromy charges $Q_{(J;J')}$ of any allowed branching function with respect to all simple currents $(J;J')$ in the identification group vanish. Here $Q_{(J;J')}((\Lambda;\Lambda')) = Q_J(\Lambda) - Q_{J'}(\Lambda')$, where $Q_J(\Lambda)$ is the combination $Q_J(\Lambda) = \Delta_\Lambda + \Delta_J - \Delta_{J\star\Lambda}$ of conformal weights. Second, the equivalence classes in the field identification are precisely the orbits of the identification group,

$$[\Lambda;\Lambda'] = \{\, (M;M') \mid M = J\star\Lambda,\ M' = J'\star\Lambda' \text{ for some } (J;J')\in G_{\rm id} \,\}\,. \tag{7.3}$$

Moreover, provided that with respect to each subgroup of $G_{\rm id}$ all orbits have a common length, the so obtained spectrum corresponds to a consistent conformal field theory. In particular, by taking one branching function out of each $G_{\rm id}$-orbit and combining them diagonally one arrives at a modular invariant partition function, and the modular S-matrix is given by $\mathcal{S}_{[\Lambda;\Lambda'],[M;M']} = S_{\Lambda,M}\,(S'_{\Lambda',M'})^*$, where $(\Lambda;\Lambda')$ and $(M;M')$ are arbitrary representatives of the orbits $[\Lambda;\Lambda']$ and $[M;M']$, respectively.

The next degree of generality, and hence difficulty, is reached when orbits with different sizes appear. (Of course, all sizes are divisors of the size N of the orbit through $(0;0)$, on which $G_{\rm id}$ acts freely, i.e. of the size of $G_{\rm id}$. Orbits of non-maximal size are referred to as *fixed points.*) In this situation, taking precisely one representative out of each orbit $[\Lambda;\Lambda']$ would not lead to a modular invariant partition function. Rather, the combination

$$Z = \sum_{\substack{[\Lambda;\Lambda']\\ Q=0}} |G_{\Lambda,\Lambda'}|\cdot|\sum_{(J;J')\in G_{\rm id}/G_{\Lambda,\Lambda'}} b_{(J;J')\star(\Lambda;\Lambda')}\,|^2 \tag{7.4}$$

is modular invariant, where the *stabilizer subgroup* $G_{\Lambda,\Lambda'}\subseteq G_{\rm id}$ consists of those elements of $G_{\rm id}$ that leave $(\Lambda;\Lambda')$ invariant. However, because of $G_{0,0}=\{id\}$, $|G_{\rm id}|^2$ copies of the vacuum

appear in (7.4), so that one would like to divide (7.4) by this factor. But this inevitably leads to fractional coefficients in the partition function which does not fit with the interpretation of the partition function as a sum of squares of characters.

A closer look at (7.4) suggests that any fixed point should in fact correspond to $|G_{\Lambda,\Lambda'}|$ many rather than to a single primary field. According to such a description, called the *resolution* of the fixed point, the labelling of fields is not by the orbit $[\Lambda;\Lambda']$ alone, but a fixed point $(\Lambda;\Lambda')$ splits into different fields labelled as $([\Lambda;\Lambda'],\Psi)$ with Ψ running from 1 to $|G_{\Lambda,\Lambda'}|$. As far as SL(2,$\mathbb{Z}$)-representations are concerned, fixed point resolution amounts to [14] decomposing the term in (7.4) that corresponds to a fixed point as the square of a sum of $|G_{\Lambda,\Lambda'}|$ functions, each of which differs from the branching function by a certain *character modification* and which is interpreted as the character of an irreducible module. There is then also a corresponding modification of the S-matrix elements that involve fixed points.

The terms 'field identification' and 'character modification' are actually misnomers. Not the fields are to be identified, but rather, several distinct combinations of highest weights must be identified because they describe one and the same field of the coset theory. And it is not the characters which get modified, but rather the branching functions have to be modified in order that they coincide with the true characters of the coset conformal field theory.

Moreover, in most cases the required modifications can be interpreted in terms of the characters and S-matrix of another (putative) conformal field theory, called the *fixed point theory.*

8 Branching spaces and the factorization property

In the description of fixed point resolution in terms of fixed point conformal field theories, part of the information about the coset theory is encoded in the fixed point theory, so that it is far from obvious how this information could possibly be obtained from data of g and g' alone. For a large class of coset theories, this puzzle has been resolved in [16]. Namely, as described in more detail below, the character modifications can be expressed as twining branching functions associated to twining characters of g and g', and the fixed point theory corresponds to the coset construction for the corresponding orbit Lie algebras $\breve{\mathsf{g}}$ and $\breve{\mathsf{g}}'$. This way one solves the fixed point resolution problem not only at the level of SL(2,$\mathbb{Z}$)-representations, but also directly at the level of the underlying modules of the chiral algebra.

The class of coset theories we have analyzed are those for which for each $(J;J')\in G_{\rm id}$ the diagram automorphisms ω of g and ω' of g' that correspond to the simple currents J and J', respectively, are related by the *factorization property*

$$\omega|_{\mathsf{g}'} = \omega' . \tag{8.1}$$

Among the theories satisfying this condition are in particular the *diagonal cosets*, for which $\bar{\mathsf{g}}$ is the direct sum $\bar{\mathsf{h}}\oplus\bar{\mathsf{h}}$ of two copies of a simple Lie algebra $\bar{\mathsf{h}}$ and $\bar{\mathsf{g}}'\cong\bar{\mathsf{h}}$ is embedded diagonally into $\bar{\mathsf{g}}$ ((8.1) also holds for generalizations where $\bar{\mathsf{h}}$ is reductive and m copies of $\bar{\mathsf{h}}$ are embedded in $n>m$ copies thereof). Also note that any two diagram automorphisms ω_1 and ω_2 which correspond to simple currents commute. We will use automorphisms ω_i that fulfill the factorization property to implement field identification; the fact that ω_1 and ω_2 commute will imply that the maps implementing field identification will commute as well.

For diagonal cosets, the embedding of the relevant affine algebras $\mathsf{g}'\cong\mathsf{h}$ into $\mathsf{g}\cong\mathsf{h}\oplus\mathsf{h}$ is given by $E_m^{\pm\bar\alpha}=E_{(1),m}^{\pm\bar\alpha}+E_{(2),m}^{\pm\bar\alpha}$, $H_m^i=H_{(1),m}^i+H_{(2),m}^i$ and $K'=K_{(1)}+K_{(2)}$. Any diagram

automorphism respects the triangular decomposition of g, i.e. $\omega(\mathsf{g}_\#) = \mathsf{g}_\#$ for $\# \in \{+,\circ,-\}$. (Conversely, any automorphism of an affine Lie algebra with this property is induced by a symmetry of the Dynkin diagram.) Also, for diagonal cosets the triangular decompositions of g and g' are compatible in the sense that $\mathsf{g}'_\# \subseteq \mathsf{g}_\#$, and in fact $\mathsf{g}'_\# = \mathsf{g}_\# \cap \mathsf{g}'$ for $\# \in \{+,\circ,-\}$. When combined with the ω-twining property (3.3) of τ_ω, the factorization property (8.1) then implies that also $\omega'(\mathsf{g}'_\#) \subseteq \mathsf{g}'_\#$.

Consider now a unitary highest weight module V of g on which the central elements $K_{(1)}$ and $K_{(2)}$ act as non-negative integer multiples $k_{(1)}$, $k_{(2)}$ of the identity. We can view V as a g'-module, on which the central element K' acts as the multiple $k' = k_{(1)} + k_{(2)}$ of the identity. The coset Virasoro algebra then acts on V via the generators $\dot{L}_m = L_{(1),m} + L_{(2),m} - L'_m$, where the individual terms are obtained via the Sugawara construction for h at levels $k^\vee_{(1)} = 2k_{(1)}/(\bar\theta\,|\,\bar\theta)$, $k^\vee_{(2)} = 2k_{(2)}/(\bar\theta\,|\,\bar\theta)$ and $k^\vee_{(1)} + k^\vee_{(2)}$, respectively. The identification group for diagonal cosets consists of all simple currents of the form $(J;J') = (k^\vee_{(1)}\Lambda, k^\vee_{(2)}\Lambda; (k^\vee_{(1)}+k^\vee_{(2)})\Lambda)$ with Λ a so-called cominimal [12] fundamental h-weight, and hence is isomorphic to the group of those symmetries of the Dynkin diagram of h which do not leave the zeroth node fixed.

Moreover, because of $\omega'(\mathsf{g}'_+) \subseteq \mathsf{g}'_+$ we have $\mathsf{g}'_+ {\cdot} (\tau_\omega(v)) = 0$ whenever $\mathsf{g}'_+ {\cdot} v = 0$; thus τ_ω maps highest weight vectors with respect to g' in L_Λ to highest weight vectors with respect to g' in $L_{\omega^\star\Lambda}$. Similarly, if v_1 and v_2 lie in one and the same irreducible g'-module, then so do $\tau_\omega(v_1)$ and $\tau_\omega(v_2)$. It follows that on any irreducible g-module L_Λ the map τ_ω factorizes into a mapping $\dot\tau_\omega$, of the same order N, which maps any irreducible g'-submodule of L_Λ to some g'-submodule of $L_{\omega^\star\Lambda}$ (i.e., a map for which, roughly speaking, only the positions of the g'-modules $L_{\ell;\Lambda'}$ and $L_{\dot\tau_\omega\ell;\omega'^\star\Lambda'}$ in the respective g-modules matter) and the map $\tau_{\omega'}$ defined via the automorphism ω' of g'. Hence, decomposing L_Λ into g'-modules as

$$L_\Lambda = \bigoplus_\ell L_{\ell;\Lambda'} \tag{8.2}$$

(thus the g'-modules in L_Λ are labelled by $\ell = 1, 2, ...$, and as a redundant information also the highest g'-weight $\Lambda' \equiv \Lambda'(\ell)$ is displayed), the analogous decomposition of $L_{\omega^\star\Lambda}$ reads

$$L_{\omega^\star\Lambda} \equiv \tau_\omega(L_\Lambda) = \bigoplus_\ell \tau_\omega(L_{\ell;\Lambda'}) = \bigoplus_\ell L_{\dot\tau_\omega\ell;\omega'^\star\Lambda'}\,. \tag{8.3}$$

In terms of the branching spaces

$$L_{\Lambda,\Lambda'} = \mathrm{span}_{\mathbb{C}}\{v_{\ell;\lambda'} \mid \lambda'(\ell) = \Lambda'\} \tag{8.4}$$

($v_{\ell;\lambda'}$ denotes the highest weight vector of $L_{\ell;\Lambda'}$), the decomposition (8.2) reads $L_\Lambda = \bigoplus_\ell L_{\ell;\Lambda'} \cong \bigoplus_{\Lambda'} L_{\Lambda,\Lambda'} \otimes L_{\Lambda'}$, where the summation is over all integrable highest weights of g' at the relevant level. The branching functions $b_{\Lambda,\Lambda'}$ appearing in (7.2) are the traces

$$b_{\Lambda,\Lambda'}(\tau) = \mathrm{tr}_{L_{\Lambda,\Lambda'}} q^{\dot L_0 - \dot C/24} \tag{8.5}$$

($q = \exp(2\pi\mathrm{i}\tau)$) over the branching spaces. From the factorization property it follows that

$$b_{\omega^\star\Lambda,\omega^{\star\prime}\Lambda'} = b_{\Lambda,\Lambda'}\,, \tag{8.6}$$

in agreement with the fact that $(\omega;\omega')$ corresponds to an identification current $(J;J')$.

Further, it also follows from (8.1) that the coset Virasoro algebra is ω-invariant,

$$\omega(\dot L_m) = \omega(L_m) - \omega'(L'_m) = \dot L_m\,. \tag{8.7}$$

Together with the ω-twining property (3.3) of $\mathring{\tau}_\omega$, (8.7) implies that $\mathring{\tau}_\omega:\ L_{\Lambda,\Lambda'} \to L_{\omega^\star\Lambda,\omega'^\star\Lambda'}$ intertwines the action of the coset Virasoro algebra, i.e. we have

$$[\mathring{\tau}_\omega, \mathring{L}_m] = 0 \tag{8.8}$$

for all $m \in \mathbb{Z}$. In particular, $L_{\Lambda,\Lambda'}$ and $L_{\omega^\star\Lambda,\omega'^\star\Lambda'}$ carry isomorphic representations of the coset Virasoro algebra. Now in order to keep the interpretation of g' as a redundancy symmetry, not only the coset Virasoro algebra, but also the full (maximally extended) chiral algebra $\mathring{\mathcal{W}}$ of the coset theory must be well-defined on g'-orbits, i.e. $\mathring{\mathcal{W}}$ must commute with g'. (As a consequence, $L_{\Lambda,\Lambda'}$ and $L_{\omega^\star\Lambda,\omega'^\star\Lambda'}$ also carry isomorphic representations of $\mathring{\mathcal{W}}$, as is already suggested (but not implied) by (8.6).) Further, compatibility with field identification requires that the intertwining property of $\mathring{\tau}_\omega$ must be valid not only for the coset Virasoro algebra, but for $\mathring{\mathcal{W}}$ as well.

9 Fixed points

If $(\Lambda;\Lambda')$ is a fixed point of an identification current, i.e. if $\omega^\star(\Lambda)=\Lambda$ and $\omega'^\star(\Lambda')=\Lambda'$, the map $\mathring{\tau}_\omega$ is an *endo*morphism, so that we can define the *twining branching function* $b^{[\omega]}_{\Lambda,\Lambda'}$ as the trace

$$b^{[\omega]}_{\Lambda,\Lambda'}(\tau) := \mathrm{tr}_{L_{\Lambda,\Lambda'}}\, \mathring{\tau}_\omega\, q^{\mathring{L}_0 - \mathring{C}/24}\,. \tag{9.1}$$

(For diagonal cosets a fixed point is a combination $(\Lambda_{(1)}, \Lambda_{(2)}; \Lambda')$ of three weights of h that are fixed with respect to the same simple current automorphism ω_h of h). Using the fact that $[\mathring{L}_m, L'_n]=0$, so that $q^{L_m} = q^{L'_m}\cdot q^{\mathring{L}_m}$, and that $[\mathring{\tau}_\omega, \mathring{L}_m]=0$, we obtain

$$\chi^{[\omega]}_\Lambda(\tau) = \sum_{\Lambda'} b^{[\omega]}_{\Lambda,\Lambda'}(\tau)\, \chi^{[\omega']}_{\Lambda'}(\tau)\,, \tag{9.2}$$

i.e. $b^{[\omega]}_{\Lambda,\Lambda'}(\tau)$ arises in the decomposition of the twining characters of g with respect to the automorphism τ_ω into twining characters of g' with respect to $\tau_{\omega'}$ (this justifies the name 'twining branching function'). In particular, analogously as for ordinary branching functions, the twining branching functions can in principle be computed from the twining characters $\chi^{[\omega]}_\Lambda$ and $\chi^{[\omega']}_{\Lambda'}$. While doing this directly would be cumbersome, one can employ the equivalence between twining characters and the characters of the orbit Lie algebras, so that $b^{[\omega]}_{\Lambda,\Lambda'}$ can be computed as the ordinary branching function

$$b^{[\omega]}_{\Lambda,\Lambda'} = \breve{b}_{\breve{\Lambda},\breve{\Lambda}'} \tag{9.3}$$

of the coset construction $\breve{\mathsf{g}}/\breve{\mathsf{g}}'$ of orbit Lie algebras. (Above only the case when $\bar{\mathsf{g}}$ and $\bar{\breve{\mathsf{g}}}$ are simple has been described explicitly; in the general case, one deals with a product of twining characters of the various summands of g respectively g'.)

As a consequence, the twining branching functions possess nice modular transformation properties. In particular, under $\tau \mapsto -1/\tau$ they transform with

$$\mathcal{S}^{[\omega]}_{(\Lambda;\Lambda'),(M;M')} = S^{[\omega]}_{\Lambda,M}\, (S^{[\omega']}_{\Lambda',M'})^* = \breve{S}_{\breve{\Lambda},\breve{M}}\, (\breve{S}'_{\breve{\Lambda}',\breve{M}'})^*\,, \tag{9.4}$$

where $S^{[\omega]}_{\lambda,\mu}$ respectively $S^{[\omega']}_{\lambda',\mu'}$ are the corresponding S-matrices for the twining characters of g respectively g', which in turn are nothing but the S-matrices of the respective orbit Lie algebras.

Also, up to an over-all phase the matrix $\mathcal{T}^{[\omega]}$ which describes the behavior under $\tau \mapsto \tau+1$ coincides with the restriction of the ordinary (untwined) T-matrix $T^{\mathsf{g}} \otimes (T^{\mathsf{g}'})^*$ to the fixed points.

It is also not difficult to see that if J is a simple current of g, then $\breve{J}$ is a simple current of $\breve{\mathsf{g}}$ (albeit possibly the trivial one, i.e. the identity field). For coset theories, an analogous statement holds for identification currents, and hence each selection rule of g/g' leads to a selection rule of $\breve{\mathsf{g}}/\breve{\mathsf{g}}'$. As a consequence, the matrix element $\mathcal{S}^{[\omega]}_{(\Lambda;\Lambda'),(M;M')}$ does not depend on the choice of representatives of the orbits of $(\Lambda;\Lambda')$ and $(M;M')$. This implies e.g. that, analogously to (8.6),

$$b^{[\omega_1]}_{\omega_2^\star\Lambda,\omega_2^\star{}'\Lambda'} = b^{[\omega_1]}_{\Lambda,\Lambda'} \,. \tag{9.5}$$

Now recall from the discussion after (8.8) that field identification requires that any element of the coset chiral algebra $\dot{\mathcal{W}}$ commutes with $\dot{\tau}_\omega$. For fixed points this implies that already the eigenspaces with respect to $\dot{\tau}_\omega$ of the branching spaces form modules over the coset Virasoro algebra, and in fact over the coset chiral algebra $\dot{\mathcal{W}}$. This way the implementation of field identification by the maps $\dot{\tau}_\omega$ enforces fixed point resolution: one has to split the branching spaces $L_{\Lambda,\Lambda'}$ into the eigenspaces with respect to the action of the stabilizer group $G_{\Lambda,\Lambda'}$. Thus the label Ψ of the resolved fixed points $([\Lambda;\Lambda'],\Psi)$ corresponds to the group characters $\Psi: G_{\Lambda,\Lambda'} \to \mathbb{C}$ of $G_{\Lambda,\Lambda'}$; each $G_{\Lambda,\Lambda'}$-character Ψ gives rise to an eigenspace $L^{(\Psi)}_{\Lambda,\Lambda'}$ of $L_{\Lambda,\Lambda'}$ on which $\dot{\tau}_\omega$ has eigenvalue $\Psi(\omega)$. Thus, denoting the (Virasoro specialized) character of the eigenspace $L^{(\Psi)}_{\Lambda,\Lambda'}$ by $b^{(\Psi)}_{\Lambda,\Lambda'}$, we have

$$b_{\Lambda,\Lambda'}(\tau) = \sum_{\Psi\in G^\star_{\Lambda,\Lambda'}} b^{(\Psi)}_{\Lambda,\Lambda'}(\tau) \,, \tag{9.6}$$

where $G^\star_{\Lambda,\Lambda'}$ is the group of characters of $G_{\Lambda,\Lambda'}$. Since the coefficients in the expansion of $b^{(\Psi)}_{\Lambda,\Lambda'}$ in powers of q count the number of eigenstates at the relevant grade, they are manifestly non-negative integers, and hence the functions $b^{(\Psi)}_{\Lambda,\Lambda'}$ can be interpreted as the characters of the resolved fixed points. Further, we have $b^{[\omega]}_{\Lambda,\Lambda'} = \sum_{\Psi\in G^\star_{\Lambda,\Lambda'}} \Psi(\omega)\, b^{(\Psi)}_{\Lambda,\Lambda'}$, which by the orthogonality of the group characters can be inverted to $b^{(\Psi)}_{\Lambda,\Lambda'} = \sum_{\omega\in G_{\Lambda,\Lambda'}} \Psi^*(\omega)\, b^{[\omega]}_{\Lambda,\Lambda'}/|G_{\Lambda,\Lambda'}|$. Of course, one still has to implement the field identification with respect to the rest of $G_{\rm id}$, which however, owing to (9.5), just amounts to considering orbits of identical twining branching functions. Thus, employing also the convention to set $\Psi(\omega)=0$ for $\Psi\in G^\star_{\Lambda,\Lambda'}$ whenever $\omega\notin G_{\Lambda,\Lambda'}$, the characters of the resolved fixed points are

$$\begin{aligned} \mathcal{X}_{[\Lambda;\Lambda'],\Psi} &= \frac{1}{|G_{\rm id}|} \sum_{\omega\in G_{\rm id}} b^{(\Psi)}_{\omega^\star\Lambda,\omega'^\star\Lambda'} = \frac{1}{|G_{\rm id}|\cdot|G_{\Lambda,\Lambda'}|} \sum_{\omega_1,\omega_2\in G_{\rm id}} \Psi^*(\omega_1)\, b^{[\omega_1]}_{\omega_2^\star\Lambda,\omega_2'{}^\star\Lambda'} \\ &= \frac{1}{|G_{\Lambda,\Lambda'}|} \sum_{\omega\in G_{\rm id}} \Psi^*(\omega)\, b^{[\omega]}_{\Lambda,\Lambda'} = b^{(\Psi)}_{\Lambda,\Lambda'} \,. \end{aligned} \tag{9.7}$$

In other words, while in the presence of fixed points the true characters of a coset conformal field theory are not the branching functions, they can be obtained as linear combinations of branching functions of the original coset theory and those for the embedding of the associated orbit Lie algebras. Also note that when combining the two chiral halves in the diagonal modular invariant, we obtain less combinations of states than one would naively expect. This is the projection alluded to in the first section.

For the S-matrix which describes the behavior of these characters under the modular transformation $\tau \mapsto -1/\tau$, we find (by first expanding $b^{[\omega]}_{\Lambda,\Lambda'}(-1/\tau)$ in terms of the $b^{[\omega]}_{M,M'}(\tau)$ and then

re-expressing the $b^{[\omega]}_{M,M'}$ through the $b^{(\Psi)}_{\Lambda,\Lambda'}$)

$$\mathcal{S}_{([\Lambda;\Lambda'],\Psi),([M;M'],\tilde\Psi)} = \frac{|G_{\rm id}|}{|G_{\Lambda,\Lambda'}|\cdot|G_{M,M'}|} \sum_{\omega\in G_{\rm id}} \Psi^*(\omega)\, \mathcal{S}^{[\omega]}_{[\Lambda;\Lambda'],[M;M']}\, \tilde\Psi(\omega)\,. \tag{9.8}$$

Note that $\Psi\in G^\star_{\Lambda,\Lambda'}$ and $\tilde\Psi\in G^\star_{M,M'}$, so that the summation is effectively only over the subgroup $G_{\Lambda,\Lambda'}\cap G_{M,M'}\subseteq G_{\rm id}$ of the identification group. Thus if either $(\Lambda;\Lambda')$ or $(M;M')$ is not a fixed point, the sum in (9.8) contains only a single term, namely the one with $\omega=id$, and the S-matrix is just a multiple of the ordinary S-matrix element for the corresponding branching functions. Similarly, the transformation under $\tau\mapsto\tau+1$ is implemented by

$$\mathcal{T}_{([\Lambda;\Lambda'],\Psi),([M;M'],\tilde\Psi)} = \delta_{[\Lambda;\Lambda'],[M;M']}\, \delta_{\Psi,\tilde\Psi}\, T_{(\Lambda;\Lambda')}\,. \tag{9.9}$$

One can check that the matrix (9.8) is symmetric and unitary, that its square gives a permutation of order two on the primary fields of the coset conformal field theory, and that $\mathcal{S}$ and $\mathcal{T}$ generate a unitary representation of SL(2,$\mathbb{Z}$).

As an illustration, consider the case $G_{\rm id}\cong\mathbb{Z}_N$ with N prime. Then $G_{\rm id}$ has N characters $\Psi=\Psi_k,\ k=0,1,\dots,N-1$, acting on $n\in\mathbb{Z}_N$ as $\Psi_k(n)=\zeta^{kn}$ with $\zeta=\exp(2\pi{\rm i}/N)$. In this case for $\omega\in G_{\rm id}$ ($\omega\neq id$) the matrix $\mathcal{S}^{[\omega^n]}$ is the same for all $n\neq 0$, $\mathcal{S}^{[\omega^n]}_{(\Lambda;\Lambda'),(M;M')} =: \mathring{\mathcal{S}}_{(\Lambda;\Lambda'),(M;M')}$, while $\mathcal{S}^{[\omega^0]}_{(\Lambda;\Lambda'),(M;M')} = \mathcal{S}_{(\Lambda;\Lambda'),(M;M')} \equiv S_{\Lambda,M}(S'_{\Lambda',M'})^*$. In the interesting case when both combinations of weights are fixed points, we then obtain

$$\mathcal{S}_{([\Lambda;\Lambda'],\Psi_k),([M;M'],\Psi_l)} = \tfrac{1}{N}\,\mathcal{S}_{(\Lambda;\Lambda'),(M;M')} + \big[\delta_{k,l}-\tfrac{1}{N}\big]\,\mathring{\mathcal{S}}_{(\Lambda;\Lambda'),(M;M')}\,, \tag{9.10}$$

which is precisely the expression for the S-matrix that had been conjectured in [14].

10 Further applications and problems

We conclude with some remarks on related issues and open questions.

- On the purely mathematical side, one may wonder whether the notion of orbit Lie algebra can still be given a meaning for diagram automorphisms which do not satisfy the linking condition (4.2), maybe as a Borcherds algebra or as a generalized Kac–Moody algebra in the sense of [2, §11.13].

Also, we believe that the dual Lie algebra of $\breve{\mathsf{g}}$, i.e. the Lie algebra obtained from $\breve{\mathsf{g}}$ by reversing the direction of the arrows of the Dynkin diagram, is precisely the subalgebra of the dual algebra of g that stays fixed under the automorphism $\tilde\omega$ of the dual Lie algebra that corresponds to ω. [7] A proof of this connection would certainly be welcome.

- In the case of untwisted affine Lie algebras, the characters of the orbit Lie algebra $\breve{\mathsf{g}}$, and hence the twining characters as well, carry a unitary representation of $SL(2,\mathbb{Z})$ whenever the diagram automorphism corresponds to a simple current, i.e. belongs to the maximal abelian normal subgroup $\mathcal{Z}(\mathsf{g})$ of the group of diagram automorphisms. This should be related to the fact [17] that the automorphisms in $\mathcal{Z}(\mathsf{g})$ are precisely the *localizable* diagram automorphisms.
- The orbit Lie algebras of untwisted affine Lie algebras correspond precisely to the fixed point theories introduced in [14], except for the cases of the order two automorphisms of $\mathsf{g}=C^{(1)}_{2n}$ or

[7] In the case of affine algebras, these dual orbit Lie algebras have been used in the reduction of Toda field equations [18, 19].

$B_{n+1}^{(1)}$, where the orbit Lie algebra is the twisted algebra $\tilde{B}_n^{(2)}$. In the latter cases so far the orbit Lie algebra could not be interpreted in terms of a genuine conformal field theory. From the results of [4] one also knows that such a conformal field theory would have to be non-unitary.

- In contrast to (generalized) diagonal cosets, for general coset theories the factorization property (8.1) does not hold. Thus in the general case one cannot restrict oneself to diagram automorphisms ω, ω' of g and g'. Rather one has to allow for more general outer automorphisms of g and g', which are the product of a diagram automorphism and some compensating inner automorphism corresponding to an element of the Weyl group. However, we expect that the generalization of our ideas to arbitrary cosets will not involve any further new representation theoretic concepts, but rather only additional technical complications coming from the fact that ω' and $\omega|_{\mathsf{g}'}$ differ by an inner automorphism. In particular, we expect that the formula (9.8) for the S-matrix of the coset theory will still be valid.

- In all cases that we have checked explicitly, the matrix $\mathcal{S}$ defined by (9.8) gives rise, via the Verlinde formula, to non-negative integral fusion coefficients. But a general proof of this property is still lacking.

- It is worth stressing that in the description of the coset construction, there are still open conceptual problems which are largely independent of the issues of field identification and fixed point resolution. Most importantly, it has not yet been proven that there exists a suitable closure $\tilde{\mathsf{U}}(\mathsf{g})$ of the universal enveloping algebra $\mathsf{U}(\mathsf{g})$ such that the algebra $\dot{\mathcal{W}}$ defined with the help of $\tilde{\mathsf{U}}(\mathsf{g})$ fulfills all requirements for a chiral algebra of a conformal field theory. A candidate for $\tilde{\mathsf{U}}(\mathsf{g})$ is the topological completion of $\mathsf{U}(\mathsf{g})$ that is described in [20].

In order to prove that $\dot{\mathcal{W}}$ is indeed the – maximally extended – chiral algebra of the coset theory, one must in particular show that (in the absence of field identification fixed points) the branching spaces are *irreducible* modules of $\dot{\mathcal{W}}$, and that each isomorphism class of irreducible modules appears precisely once. (Note that as modules over the coset Virasoro algebra, the branching spaces are highly reducible as soon as $\dot{c} \geq 1$.) In the presence of fixed points, one has to prove the same statement for the eigenspaces of the maps $\dot{\tau}_\omega$ (in principle a further splitting of these eigenspaces could be necessary, but we believe that they are in fact already irreducible modules over $\dot{\mathcal{W}}$).

Note that according to our construction the candidate coset chiral algebra $\dot{\mathcal{W}}$ is *not* obtained as the commutant $\mathcal{W}_{\mathsf{g},\mathsf{g}'}$ of g' in $\tilde{\mathsf{U}}(\mathsf{g})$, but rather as the subalgebra of $\mathcal{W}_{\mathsf{g},\mathsf{g}'}$ that is fixed by all automorphisms yielding field identifications; this reflects the fact that not only the action of the currents of g', but also the action of the maps $\dot{\tau}_\omega$ connects different representatives of one and the same physical state. More specifically, for a field identification automorphism ω the factorization property implies that $\omega(\mathcal{W}_{\mathsf{g},\mathsf{g}'}) \subseteq \mathcal{W}_{\mathsf{g},\mathsf{g}'}$, so that ω induces an order-N automorphism of $\mathcal{W}_{\mathsf{g},\mathsf{g}'}$ and we can decompose $\mathcal{W}_{\mathsf{g},\mathsf{g}'}$ into its eigenspaces with respect to ω. The eigenspace $\mathcal{W}_{\mathsf{g},\mathsf{g}'}^{(0)}$ that is left invariant under all these automorphisms is a subalgebra of $\mathcal{W}_{\mathsf{g},\mathsf{g}'}$ and contains all elements of $\dot{\mathcal{W}}$ that are intertwined by $\dot{\tau}_\omega$ (in particular, according to (6.3), the coset Virasoro algebra). The fact that field identification is implemented by the maps $\dot{\tau}_\omega$ then implies that $\dot{\mathcal{W}} = \mathcal{W}_{\mathsf{g},\mathsf{g}'}^{(0)}$.

- The interpretation of g' as a gauge symmetry is a basic ingredient in the geometric approach to coset theories via WZW sigma models. There are indications [21] that in this formulation fixed point resolution can be implemented by considering also topologically non-trivial G'-principal bundles, where G' is the relevant Lie group whose Lie algebra is g'. Our results therefore suggest interesting new relations between these bundles in different topological sectors.

Twining characters and orbit Lie algebras are natural structures associated to diagram automorphisms of symmetrizable Kac–Moody algebras. We believe that they should be useful in various different situations where such automorphisms play a rôle. As for the case of conformal field theory, the following applications come to mind: the explanation of the fixed point resolution that is present [4] in integer spin simple current invariants, the influence of fixed points on the relation [22] between conformal field theory and graphs, and the computation of the generalized S-matrices $S(J)$ that were defined in [23].

Finally let us remark on the issue of fixed point resolution in more general situations. The fact that we could attribute the presence of different orbit lengths to the action of a *finite* group of redundancy symmetries certainly constitutes a significant simplification. In particular, it enabled us to describe fixed point resolution in terms of the eigenspaces of these redundancy symmetries. This will typically no longer be possible in the general case.
Another major advantage of the theories considered here is that it is easy to see that of a naive implementation of the redundancies according to (1.1) leads to inconsistencies. In other contexts such inconsistencies, if present, will be much more difficult to detect.

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